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Representation of lattices by fuzzy weak congruence relations ☆

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Abstract

Fuzzy (lattice valued) weak congruences of abstract algebras are investigated. For an algebra, the family of all such fuzzy relations is a complete lattice; its structure and cut properties are investigated and fully described. These fuzzy weak congruences are applied in representation of complete and algebraic lattices. A wider class of lattices can be represented in such a fuzzy framework, than in classical algebra. We prove that there is a straightforward representation of any complete lattice, using it as a co-domain. In a more general case, it is proved that several subdirect powers of lattices are also representable by fuzzy weak congruences.

Keywords: Fuzzy relations; Algebra; Fuzzy algebra; Fuzzy weak congruence; Complete lattice

1. Introduction

We deal with the fuzzy approach to congruences in general algebra, where the set of membership values is a complete lattice (an approach started by Goguen in [18]). Investigating these fuzzy compatible relations on algebraic structures, we apply our results to the classical lattice theory, namely in representation of lattices.

Our link to existing results in both fuzzy and classical algebra is as follows. It is known that algebraic structures, like other notions in the fuzzy framework, depend on the membership function. This function offers a variety of properties and possibilities for applications, much more than the classical characteristic function. In addition, in the cut-worthy approach, there is a connection between a fuzzy structure as a mapping and a collection of crisp sub-structures of the same domain. Therefore, these investigations have a long history. At the beginning there were fuzzy groups (Rosenfeld [28] and Das [13], later Demirci [14] and others), then other structures and applications (Malik and Mordeson e.g., [25]). Investigations of notions from universal algebra were also carried out (e.g., Di Nola, Gerla

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[15] and the present authors [5,30–34]). In particular, fuzzy congruences, which are the central topic here, have been widely investigated (e.g., Kim and Bae [20], Kuroki [23], Kuraoka and Suzuki [22], Tan [39], Murali [26], recently Louskou-Bozapalidou [24], Ignjatovic et al. [19]), and also lattices of fuzzy congruences (Ajmal and Thomas [1] and others, recently Rainard and Mangalambal [27]; for hyper-structures Bakhshi and Borzooei [2] and Cabrera et al. [7,8]). Concerning our approach to fuzzy relations on fuzzy sets, it appeared in Filep's paper [17]. A version of weak reflexivity was used long time ago in the paper [40] by Yeh and Bang, and then also in the paper [37].

In fuzzy algebraic investigations the set of membership values is the unit interval, or more generally a complete lattice (in many cases residuated); posets or relational systems are also used [30]. In the recent period, Bělohlávek (also together with Vychodil) develops and investigates the most important universal algebraic topics (congruences, subalgebras, products, and also varieties, see books [3,4]). The set of values of fuzzy structures in this approach is usually a complete residuated lattice. Let us mention that the residuation in the structure of multilattice was introduced in [9].

Our motivation is connected to our investigations of weak congruences in universal algebra, (see e.g. papers [10,11,16,29,36], and monograph [35]). Namely, the lattice of all weak congruences of an algebra is algebraic; congruence lattices of all its subalgebras, as well as, up to an isomorphism, the lattice of its subalgebras, come out to be its sublattices. An open problem in universal algebra is a representation of algebraic lattices by weak congruences. Here we investigate the analogue topic in the fuzzy framework. Using the idea from [37], we introduce fuzzy (lattice valued) weak congruences of an algebra and connect them with congruences on its fuzzy subalgebras. We prove that the lattice of fuzzy weak congruences is complete and fully describe its structure in lattice-theoretic terms. In addition, we deal with the representation problem of lattices by fuzzy weak congruences. It turns out that in this fuzzy framework it is possible to represent complete lattices which are not algebraic. We also prove representation theorem for several classes of lattices which are known to be non-representable in the classical way.

2. Preliminaries

We deal with fuzzy algebraic structures whose co-domain is a complete lattice. Here we list relevant notions and notation. For theoretical background in lattice theory and universal algebra, we refer to books [12] and [6].

2.1. Algebras, lattices

An **algebra** (universal algebra) is a pair (A, F), denoted by $\mathcal{A}$, where A is a nonempty set and F is a set of operations on A. A **subalgebra** of $\mathcal{A}$ is an algebra defined on a subset $B \subseteq A$, where the operations are restrictions of the operations from F. A **subuniverse** of $\mathcal{A}$ is a subset (which may be empty) closed under operations.

An equivalence relation ρ on A which is compatible with all the operations from F, in the sense that $x_i \rho y_i$, $i = 1, \dots, n$ imply $f(x_1, \dots, x_n) \rho f(y_1, \dots, y_n)$, is a **congruence** on $\mathcal{A}$.

If $\mathcal{A}$, $\mathcal{B}$ are algebras of the same type, then the mapping $\varphi : A \to B$ is a **homomorphism** from A to B if for every n-ary $f \in F$ and all $x_1, \dots, x_n \in A$, $\varphi(f(x_1, \dots, x_n)) = f(\varphi(x_1), \dots, \varphi(x_n))$; for a nullary operation c, $f(c_A) = c_B$. If $\mathcal{A} = \mathcal{B}$, φ is an **endomorphism**.

Our main notion is a **complete lattice**, denoted by $(L, \wedge, \vee, \leqslant)$. A greatest, top element is denoted by 1, and a smallest, bottom element by 0. We use also the notion of a **principal filter** generated by $a \in L$, denoted by $\uparrow a$: $\uparrow a = \{x \in L \mid a \leqslant x\}$. A **principal ideal** generated by a is defined dually: $\downarrow a = \{x \in L \mid x \leqslant a\}$. For $a, b \in L$, $a \leqslant b$ the **interval** $[a, b]$ is defined by: $[a, b] = \uparrow a \cap \downarrow b$. A sublattice N of L is **convex**, if $a, b \in N$ implies $[a, b] \subseteq N$. An element a of the lattice L is said to be **codistributive** if for all $x, y \in L$

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y).$$

Proposition 1. *If a is a codistributive element in L, then $x \mapsto a \wedge x$ is an endomorphism of L onto $\downarrow a$.* □

By the above, if a is a codistributive element in L, then the sublattice $\downarrow a$ is a **retraction** in L (i.e., it is a sublattice and at the same time a homomorphic image of L in a homomorphism which fixes it).

A complete lattice L is **algebraic** if it is compactly generated, i.e., if every element in L is supremum of compact elements (see [6]).

The following are known facts about complete lattices and their role in universal algebra (see e.g. [6]).