Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 257 (2014) 169–183
FUZZY sets and systems
www.elsevier.com/locate/fss

Solution of linear differential equations with fuzzy boundary values

Nizami Gasilov [a], Şahin Emrah Amrahov [b,*], Afet Golayoglu Fatullayev [a]

[a] *Baskent University, Ankara 06810, Turkey*
[b] *Computer Engineering Department, Ankara University, Ankara, Turkey*

Available online 22 August 2013

Abstract

We investigate linear differential equations with boundary values expressed by fuzzy numbers. In contrast to most approaches, which search for a fuzzy-valued function as the solution, we search for a fuzzy set of real functions as the solution. We define a real function as an element of the solution set if it satisfies the differential equation and its boundary values are in intervals determined by the corresponding fuzzy numbers. The membership degree of the real function is defined as the lowest value among membership degrees of its boundary values in the corresponding fuzzy sets. To find the fuzzy solution, we use a method based on the properties of linear transformations. We show that the fuzzy problem has a unique solution if the corresponding crisp problem has a unique solution. We prove that if the boundary values are triangular fuzzy numbers, then the value of the solution at a given time is also a triangular fuzzy number. The defined solution is the same as one of the solutions obtained by Zadeh's extension principle. For a second-order differential equation with constant coefficients, the solution is expressed in analytical form. Examples are given to describe the proposed approach and to compare it to a method that uses the generalized Hukuhara derivative, which demonstrates the advantages of our method.

Keywords: Fuzzy boundary value problem; Fuzzy differential equation; Fuzzy set; Linear transformation

1. Introduction

Approaches to fuzzy boundary value problems (FBVPs) and other fuzzy problems can be of three types. The first approach assumes that even if only the boundary values are fuzzy, the solution is a fuzzy function and consequently the derivatives in the differential equation must be considered as fuzzy derivatives. These can be either Hukuhara or generalized derivatives. Bede demonstrated that a large class of BVPs have no solution if the Hukuhara derivative is used [4]. To overcome this difficulty, the concept of a generalized derivative was developed [5,12] and fuzzy differential equations were investigated using this concept [7,11,13,29]. Khastan and Nieto found solutions for a large enough class of boundary value problems using the generalized derivative [20]. However, their examples and examples presented here show that these solutions are difficult to interpret, because the four different problems obtained using the generalized first and second derivatives often do not reflect the nature of the problem. Liu reduced these four problems to two when the right-hand-side function is monotonic [23].

* Corresponding author. Tel.: +90 312 3800026x188.
E-mail address: emrah@eng.ankara.edu.tr (Ş.E. Amrahov).

http://dx.doi.org/10.1016/j.fss.2013.08.008

Khastan et al. demonstrated that a class of first-order linear differential equations subject to periodic boundary conditions can be solved by switching between two types of generalized derivative over the time domain [21]. Rodríguez-López improved this result for equations whose coefficients may change their sign finitely many times [28]. Using novel generalizations of the Hukuhara difference for fuzzy sets, Bede and Stefanini investigated new generalized differentiability concepts for fuzzy-valued functions [8].

To avoid difficulties with fuzzy derivatives, some researchers propose that an equivalent fuzzy integral equation should be solved instead of a fuzzy differential equation [1,14,27].

The second approach is based on Zadeh's extension principle. For a fuzzy initial value problem, the associated crisp problem is solved and in the solution the initial fuzzy value is substituted instead of the real constant. In the final solution, arithmetic operations are considered to be operations on fuzzy numbers [9,10]. To find an exact solution to the periodic boundary value problem for a first-order linear fuzzy differential equation with impulses, Nieto et al. used the crisp solution [26].

In the third approach, the fuzzy problem is transformed to a crisp problem. There are two ways to realize this approach. The first, suggested by Hüllermeier [18], uses the concept of differential inclusion. In this way, by taking an α-cut of the initial value and the solution, the given differential equation is converted to a differential inclusion and the solution is accepted as the α-cut of the fuzzy solution. Li et al. used this approach in a recent study [22]. In the paper the concept of big solutions is introduced and some existence and uniqueness theorems are obtained. Misukoshi et al. [24] have proved that, under certain conditions, this approach is equivalent to the second one for the initial value problem. The second way is offered by Gasilov et al. [15]. In this way the fuzzy problem is considered to be a set of crisp problems.

In this study, we investigate a differential equation with fuzzy boundary values. We interpret the problem as a set of crisp problems. For linear equations, we propose a method based on the properties of linear transformations. We show that, if the solution of the corresponding crisp problem exists and is unique, the fuzzy problem also has unique solution. Moreover, we prove that if the boundary values are triangular fuzzy numbers, then the value of the solution is a triangular fuzzy number at each time. We explain the proposed method on examples. We find analytical expression for solution of second-order linear differential equation with constant coefficients. We demonstrate with an example the advantages of the proposed method over the method which uses the generalized Hukuhara derivative.

The remainder of the paper is organized as follows. Section 2 provides preliminary information. In Section 3 we describe the fuzzy boundary value problem and the solution concept. The methodology is proposed in Section 4. We apply this methodology to solve examples in Section 5. In Section 6 we compare our approach with a method that uses a generalized Hukuhara derivative. Section 7 concludes and identifies further research directions.

2. Preliminaries

2.1. Basic concepts of fuzzy set theory

The notion of a fuzzy set is an extension of the classical notion of a set. In classical set theory, an element either belongs or does not belong to a given set. By contrast, in fuzzy set theory, an element has a degree of membership, which is a real number from [0, 1], in a given fuzzy set. In fuzzy set theory, classical sets are usually called *crisp* sets.

A *fuzzy set* $\widetilde{A}$ can be defined as a pair of the *universal set* U and the *membership function* $\mu : U \to [0, 1]$. If the universal set U is fixed, a membership function fully determines a fuzzy set. We denote the membership function as $\mu_{\widetilde{A}}$ to emphasize that the fuzzy set $\widetilde{A}$ is under consideration.

For each $x \in U$, the number $\mu_{\widetilde{A}}(x)$ is called the *membership degree* of x in $\widetilde{A}$.

The *support* of $\widetilde{A}$ is a crisp set and is defined as $\operatorname{supp}(\widetilde{A}) = \{x \in U \mid \mu_{\widetilde{A}}(x) > 0\}$.

Let $U = R$, where R is the set of real numbers. Let a, c, and b be real numbers such that $a < c < b$. A set $\widetilde{u}$ with membership function

$$\mu(x) = \begin{cases} \frac{x-a}{c-a}, & a \leqslant x \leqslant c \\ \frac{x-b}{c-b}, & c \leqslant x \leqslant b \\ 0, & \text{otherwise} \end{cases}$$