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Some applications of the study of the image of a fuzzy number: Countable fuzzy numbers, operations, regression and a specificity-type ordering

A. Roldán [a,*], J. Martínez-Moreno [a], C. Roldán [b]

[a] *Department of Mathematics, University of Jaén, Campus las Lagunillas s/n, 23071 Jaén, Spain*
[b] *Department of Statistics and Operations Research, University of Granada, Campus Fuentenueva s/n, E-18071 Granada, Spain*

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Abstract

In this paper, we present a characterization of the image of a fuzzy number in terms of the extremes of its level sets. Then we deduce some consequences: we show some relationships between the image of two fuzzy numbers and the images of their usual operations and it is proved that the continuity of some fuzzy numbers is preserved under the usual operations. As applications, we introduce a fuzzy regression model involving a parametric family of fuzzy semidistances between fuzzy numbers with finite image and an ordering process of fuzzy numbers with compact support that satisfies specificity-type properties.

Keywords: Fuzzy number; Discrete fuzzy number; Fuzzy regression; Operations

1. Introduction

In many real-life situations related to Economics, Business, Social Sciences, Technology, Environmental Sciences, etc., the information associated with some random experiments is imprecise. The language we use is usually full of inaccuracies that often are necessary in order to communicate (this item costs around \$100). Our measurements are imperfect since they depend on the precision of our measurement tool. At other times, data collected in surveys have a certain degree of uncertainty (a political party will get between 10 and 12 members of the Parliament in the next elections). The theory that best approaches to this degree of imprecision/uncertainty from a rigorous point of view is the theory of fuzzy sets (introduced in 1965 by Zadeh [47]), which provides a useful tool for modeling these situations.

In 1972, Chang and Zadeh [6] introduced the conception of *fuzzy number* (FN) with the consideration of the properties of probability functions. Since then, the theory of FNs and its applications have expansively been developed (such as [8,9,11,15,18,25] to name a few) in data analysis, artificial intelligence and decision making. In this field, ordering FNs and their comparison play a key role in decision-making procedures and there exist a huge number of published researches. FNs allow us to make the mathematical model of linguistic expressions such as *many*, *at least* or *about*, fuzzy quantifiers or fuzzy cardinality, many of which are not easy to fix and handle in practical manipulations.

* Corresponding author. Tel.: +34 953 211910.
E-mail addresses: afroldan@ujaen.es (A. Roldán), jmmoreno@ujaen.es (J. Martínez-Moreno), iroldan@ugr.es (C. Roldán).

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Real numbers can be seen as FNs. Mizumoto and Tanaka [29] introduced a way to operate with FNs that also extend the usual operations with real numbers. However, some difficulties are found when we try to use this definition. In practice, the interval arithmetic has also proven to be a powerful method when operating with FNs. In 2001, Voxman [41] introduced the conception of *discrete FN* (which is useful in some applications) assuming that its support is finite and gave out a canonical representation. Note that his notion of discrete FN is not an FN in the sense that we will use. Since then, a lot of mathematicians have been studying on discrete FNs and have obtained many results (for instance, see [4,40]).

Due to the increasing interest in this area and motivated by its possible applications, in this paper we introduce a slightly notion of *discrete FN* assuming that its image is a finite or countable set. This kind of FNs is interesting since, in many cases, the usual computation of FNs is only referred to certain data (a finite approximation of an FN) and, in practice, the most of examples of fuzzy structures (probabilistic metric spaces, fuzzy metric spaces in several senses and intuitionistic fuzzy metric spaces) are constructed using these classes of FNs. We prove that the family of all FNs whose image is a countable (or finite) set is closed under the usual operations (notice that, in general, arithmetic operations between discrete FNs in the sense of Voxman [41] do not preserve the closeness of the operations when we use Zadeh's Extension Principle). For this purpose, a previous study of the image of an FN (characterizing it in terms of the extremes of its level sets) is essential. Then we present some consequences of this result. On the one hand, we show some relationship between the image of two FNs and the images of their usual operations. On the other hand, we highlight that the continuity of some FNs is a property preserved under sums and differences. Finally, we develop some applications. A fuzzy regression methodology using a fuzzy semidistance between FNs with finite image is considered. This process can be useful when a few level sets of data FNs are known (for instance, when collecting data is expensive and time consuming) or when we only want to do an estimation of some level sets of the fuzzy response variable. We also describe a specificity-type method for ordering fuzzy numbers.

This paper is organized as follows. Some preliminaries and notations concerning FNs and operations between them are gathered in Section 2. In Section 3, we present our main result in order to describe the image of an FN and we deduce some consequences. Section 4 is devoted to explain some applications. Finally, Section 5 includes some concluding remarks and prospects for further work.

2. Preliminaries

Let $\mathbb{R}$ denote the set of all real numbers and $\overline{\mathbb{R}} = [-\infty, \infty]$ the extended real line (for simplicity, $+\infty$ will be denoted as ∞). Henceforth, $x_0 \in \mathbb{R}$ will be a real number.

Definition 1. Let $f : X \to Y$ be a mapping. We say that f is a *finite* (respectively, *countable*) *mapping* if its image $\operatorname{Im} f$ is a finite (respectively, countable) subset of Y.

We point out that we will use the term *countable* admitting the possibility that $\operatorname{Im} f$ is *finite*.

Definition 2. A *fuzzy set on* $\mathbb{R}$ is a map $\mathcal{A} : \mathbb{R} \to [0, 1]$. A *fuzzy number on* $\mathbb{R}$ (hereinafter, *FN*) is a fuzzy set $\mathcal{A}$ on $\mathbb{R}$ that verifies the following properties:

(1) Normality: there exists a real number $x_0 \in \mathbb{R}$ such that $\mathcal{A}(x_0) = 1$.
(2) For all $\alpha \in]0, 1]$, the set $\mathcal{A}_{[\alpha]} = \{x \in \mathbb{R}\colon\ \mathcal{A}(x) \geqslant \alpha\}$ is a closed subinterval of $\mathbb{R}$.

The set $\mathcal{A}_{[\alpha]}$ is known as the *α-level set* (or *α-cut*) of $\mathcal{A}$. The *kernel* of an FN $\mathcal{A}$ is $\ker \mathcal{A} = \mathcal{A}_{[1]}$ and its *support* is the closure $\operatorname{supp}(\mathcal{A}) = \overline{\{x \in \mathbb{R}\colon\ \mathcal{A}(x) > 0\}}$. Let $\mathcal{F}$ be the family of all FNs.

From now on, we will denote by *FFN* (respectively, by *CFN*) the family of all FNs whose image is finite (respectively, countable).

Remark 3. It is important not to confuse an FFN (whose image is finite) with a *discrete* FN in the sense of Voxman (see [4,5,41]), whose domain and image are finite.