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Decomposition approaches to integration without a measure

Salvatore Greco [a,b], Radko Mesiar [c,d], Fabio Rindone [a,*], Ladislav Sipeky [c]

[a] *Department of Economics and Business, 95029 Catania, Italy*

[b] *University of Portsmouth, Portsmouth Business School, Centre of Operations Research and Logistics (CORL), Richmond Building, Portland Street, Portsmouth PO1 3DE, United Kingdom*

[c] *Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia*

[d] *Institute of Theory of Information and Automation, Czech Academy of Sciences, Prague, Czech Republic*

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Abstract

Extending the idea of Even and Lehrer (2014) [3], we discuss a general approach to integration based on a given decomposition system equipped with a weighting function, and a decomposition of the integrated function. We distinguish two type of decompositions: sub-decomposition based integrals (in economics linked with optimization problems to maximize the possible profit) and super-decomposition based integrals (linked with costs minimization). We provide several examples (both theoretical and realistic) to stress that our approach generalizes that of Even and Lehrer (2014) [3] and also covers problems of linear programming and combinatorial optimization. Finally, we introduce some new types of integrals related to optimization tasks.

Keywords: Decomposition integral; Weighting function; Choquet integral; Optimization problems; Decision making

1. Introduction

The idea of decomposition of the integrated function f for the integration purposes is a basic feature of constructions/definitions of integrals since ever. Recall, e.g., Eudoxus of Cnidus (408–355 BC) exhaustion principle, Riemann and Lebesgue integrals (lower and upper integral sums), etc. Integration always merges two sources of information, the integrated function and weights of special functions used for decomposition purposes (e.g., measures assigning weights to sets, i.e., to characterize functions of sets), into a single representative value. In this contribution, we will deal with non-negative (measurable) functions and non-negative weights only, supposing always the monotonicity of the considered weights, and vanishing of such weights for null functions. Both from transparency of our ideas as well as for the application purposes in economics and multicriteria decision support,

* Corresponding author.
E-mail addresses: salgreco@unict.it (S. Greco), radko.mesiar@stuba.sk (R. Mesiar), frindone@unict.it (F. Rindone), ladislav.sipeky@stuba.sk (L. Sipeky).

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we will always deal with a fixed finite space $N = \{1, \dots, n\}$, where $n \in \mathbb{N}$ is a fixed positive integer. Then the power set 2^N being considered excludes any measurability constraints. Each function $f : N \to [0, \infty[= \mathbb{R}_+$ can be seen as an n-dimensional vector $\boldsymbol{x} \in \mathbb{R}_+^n$, $\boldsymbol{x} = (x_1, \dots, x_n) = (f(1), \dots, f(n))$. The aim of this contribution is a proposal of a general approach to decomposition based integration, distinguishing sub-decompositions and super-decompositions. We will stress several integrals known from the literature as particular instances of our approach. Moreover, several new types of integrals related to optimization tasks will be introduced and exemplified. The paper is organized as follows. In Section 2 we propose the idea of sub-decomposition based integrals and, similarly, super-decomposition approach to integration is discussed in Section 3. We provide several examples of application of decomposition integrals, both theoretical as well as realistic. In Section 4 we confront our approach with previous research in literature, especially with the idea of Even and Lehrer [3]. Particular decomposition based integrals are discussed in Section 5. Finally, some concluding remarks and formal proposal for future researches are added in Section 6.

2. Sub-decomposition based integrals

Any finite system of vectors of $\mathbb{R}_+^n$, $(\boldsymbol{x}^i)_{i=1}^k = (\boldsymbol{x}^1, \dots, \boldsymbol{x}^k) \in (\mathbb{R}_+^n)^k$ with $k \in \mathbb{N}$, is called a *collection*, and the set of all collections is $\mathcal{R}_n = \cup_{k \in \mathbb{N}} (\mathbb{R}_+^n)^k$. A *decomposition system* is any $\mathcal{D} \subseteq \mathcal{R}_n$ such that there exists $\boldsymbol{x} \neq \boldsymbol{0} = (0, \dots, 0)$ with $\boldsymbol{x} \in (\boldsymbol{x}^i)_{i=1}^k$ for some collection $(\boldsymbol{x}^i)_{i=1}^k \in \mathcal{D}$.

As usual, for any two $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^n$ with $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$, we write $\boldsymbol{x} \leq \boldsymbol{y}$ whenever $x_i \leq y_i$ for all $i = 1, \dots, n$.

Given a decomposition system $\mathcal{D}$, we denote

$$\tilde{\mathcal{D}} = \{\boldsymbol{x} \in \mathbb{R}_+^n \mid \boldsymbol{x} \in (\boldsymbol{x}^i)_{i=1}^k \text{ for some collection } (\boldsymbol{x}^i)_{i=1}^k \in \mathcal{D}\}.$$

Conversely, for any $X \subseteq \mathbb{R}_+^n$, with X containing at least one non-zero vector, we define

$$\mathcal{D}_X = \{(\boldsymbol{x}^i)_{i=1}^k \in \mathcal{R}_n \mid \boldsymbol{x}^i \in X \text{ for all } i = 1, \dots, k\}$$

as the complete decomposition system generated by X, and clearly $\tilde{\mathcal{D}}_X = X$ and, moreover, $\mathcal{D}_X$ is the union of all decomposition systems $\mathcal{D}$ such that $\tilde{\mathcal{D}} = X$.

Definition 1. Let $\mathcal{D}$ be a decomposition system. A mapping $A : \tilde{\mathcal{D}} \to \mathbb{R}_+$ is called a weighting function on $\mathcal{D}$ whenever

- $A(\boldsymbol{x}) \leq A(\boldsymbol{y})$ if $\boldsymbol{x} \leq \boldsymbol{y}$, $\boldsymbol{x}, \boldsymbol{y} \in \tilde{\mathcal{D}}$ (monotonicity),
- $A(\boldsymbol{x}) > 0$ for some $\boldsymbol{x} \in \tilde{\mathcal{D}}$ and $A(\boldsymbol{0}) = 0$ whenever $\boldsymbol{0} \in \tilde{\mathcal{D}}$ (boundary conditions).

Observe that if $\tilde{\mathcal{D}} = \mathbb{R}_+^n$, then any weighting function A can be seen as an aggregation function (in the sense of [5], with related boundary condition, i.e., $\sup\{A(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{R}_+^n\} = +\infty$ replaced by $\sup\{A(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{R}_+^n\} > 0$).

The following example is inspired by Even and Lehrer [3, example in Section 2].

Example 1. Consider two different work agencies $\mathcal{A}_1$ and $\mathcal{A}_2$. Each agency provides a couple of workers with exactly the same skills. However, each of the four workers can work alone, or together with one or more partners. The possible teams are identified with

$$\mathcal{T} = \{0, 1, 2\}^2 \setminus \{(0, 0)\} \subseteq \mathbb{N}_0^2,$$

where $(1, 0)$, $(0, 1)$ represent basic teams formed by a single worker from agency $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, while, e.g., $(2, 1)$ is the team formed by the two workers from $\mathcal{A}_1$ and one indifferently chosen from $\mathcal{A}_2$. Suppose we know the efficiency of each team, measured in some work unit, given by the weighting function $E : \mathcal{T} \to \mathbb{R}_+$: