Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 287 (2016) 48–56

FUZZY sets and systems

www.elsevier.com/locate/fss

On convergence of topological aggregation functions

M. Stehlík [a,b,*]

[a] *Department of Applied Statistics, Johannes Kepler University in Linz, Austria*
[b] *Institute of Statistics, University of Valparaíso, Valparaíso, Chile*

Received 23 February 2014; received in revised form 25 November 2014; accepted 26 November 2014

Available online 3 December 2014

Abstract

The concept of information aggregation is used in all scientific areas. More metric properties than required are often used when working with the mathematical concept of aggregation. This often gives the advantage of simplicity, but can sometimes obscure clues that might lead to a new discovery. Imprecision in space irregularities can be addressed by relaxing the metrizability condition using aggregation in a topological space. We study the convergence of nets of topological aggregation functions and their theoretical properties, and, in particular, the sensitivity of topological aggregation to input errors. It is clear that the underlying topology plays a crucial role in determining the aggregation properties.

Keywords: Aggregation; Semi-continuity; Strong convergence; Topology

1. Introduction

In this paper, we aim to provide a basis for topological aggregation and focus on convergence of nets of aggregation operators. In particular, we focus on the properties of the limit aggregation operator, and the related problem of interchanging the limits. Aggregation means the process of combining and merging several (e.g., numerical or geometrical) objects, see [6] and the references it contains. When working with mathematical objects and notions, we often use more metric properties than are needed; this can give the advantage of simplicity, but can sometimes have negative consequences. Using aggregation in a topological space can allow the imprecision of space irregularities to be addressed conveniently by relaxing the metrizability condition. This may play a crucial role when making inferences in fuzzy logic. Despite spatial imprecision and deformations, the other reason for introducing topological aggregation is the simple fact that, in many practical situations (e.g., cancer research, see [18], osteoporosis research, see [13], and insurance, see [5]) the aggregation is strongly related to the geometry of the underlying set, which can be strictly non-Euclidean (e.g., fractal). Topological aggregation is, therefore, a remedy for unusual statistical behavior (e.g., likelihood ratio statistics in the case of [13]) in non-Euclidean sampling spaces. This is possible because

* Correspondence to: Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile. Tel.: +56 32 2654680, +43 73224686808; fax: +43 73224686800.

E-mail addresses: Milan.Stehlik@jku.at, Milan.Stehlik@usm.cl.

http://dx.doi.org/10.1016/j.fss.2014.11.022
0165-0114/

the topological approach allows the variability necessary for regularization, and it better uses the geometry of the set. Further research will certainly be needed, but to illustrate this, in [13], the likelihood ratio statistics are regularized by scaling them relative to the fractal set. Even though we could still use classical aggregation data inputs in many situations, the optimal statistical design will change dramatically because of radical changes in the geometry of the sampling space. Therefore, we consider topological aggregation to be the most versatile aggregation that takes the geometry of the underlying set into account.

In this paper, we will demonstrate the possibilities for the topological modeling of aggregation functions, which may allow new properties to be discovered (e.g. see Example 2, where Lipschitzianity of a projection aggregation function cannot be automatically generalized to abstract space). For another example, the pointwise convergence of the Lipschitz aggregation function may increase the Lipschitzian constant. We will introduce aggregation functions in a general topological space, for which there are strong motivations, such as in insurance, cancer research, and astrophysical research into the homogeneity of galaxy fractals. Sets with a rather general nature are popular in empirical approaches, so we will introduce a theoretical framework for estimating an aggregation function in abstract sets. In Section 2 we will introduce aggregation functions in a topological space and discuss their properties. We will study Lipschitzianity, continuity, and convergence of aggregation functions in a topological space. The proofs are presented in Appendix A, to maintain the continuity of the explanation.

2. Aggregation in topological spaces

An aggregation function is a function that is applied to a set of values and returns a single, aggregated value. 1-Lipschitz aggregation functions on metric spaces have been studied previously, for example, by [10]. A binary aggregation function is a function $A : [0, 1]^2 \to [0, 1]$ that is non-decreasing in each component and for which $A(0, 0) = 0$ and $A(1, 1) = 1$.

Examples of 1-Lipschitz aggregation functions are the arithmetic mean, the product, the minimum, the maximum, weighted means, copulas, and quasicopulas. Here, we introduce an aggregation function and a Lipschitz aggregation function on topological space. It is necessary to aggregate in a more abstract way than just at intervals of [0, 1].

Definition 1. $(X, T, \leq)$ is a topological space with partial ordering $\leq$, and T is the family of the open subsets of X. An aggregation function in X^2 is a function $A : X^2 \to X$ such that

(i) it is non-decreasing in each variable, and
(ii) for each $u \in X$ there is $(x_1, x_2), (y_1, y_2) \in X^2$ such that $A(x_1, x_2) \leq u \leq A(y_1, y_2)$.

Remark 1. Notice that we do not need X to be the topological space to define an aggregation function in Definition 1, but we restricted X to be a topological space because of the main focus of this paper. Also notice that (ii) is the same as 1) in [14]. If X is a bounded chain, with top element 1 and bottom element 0, then (ii) is equivalent to the idempotency of 0 and 1. For more discussion on other types of scale see [14]. Definition 1 can easily be generalized to define an n-ary aggregation function.

Remark 2. Notice that, as mentioned also by the referee, the whole work is done in the context of general mappings $A : X^2 \to X$, where X is a topological space, sometimes with additional properties.

The Lipschitz function on a general topological space was introduced in [12]. Next, we will define the Lipschitz aggregation function on a topological space. Before presenting the formal definition, we will recall some topological concepts (see [4, pp. 302–303]). If p is an open cover of a topological space (X, T), and $x \in X$, the star of the point x is defined as

$$st_p(x) = \bigcup_{V \in p, x \in V} V.$$

By induction we can define $st_p^1(x) := st_p(x)$ and for $n \geq 2$ the n-star of x by

$$st_p^n(x) = \bigcup_{V \in p, V \cap st_p^{n-1}(x) \neq \emptyset} V.$$