Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 287 (2016) 137–153

FUZZY sets and systems

www.elsevier.com/locate/fss

A characterization of a class of uninorms with continuous underlying operators

Paweł Drygaś [a], Daniel Ruiz-Aguilera [b,*], Joan Torrens [b]

[a] *Interdisciplinary Centre for Computational Modelling, University of Rzeszów, Pigonia 1, 35-310 Rzeszów, Poland*
[b] *Department of Mathematics and Computer Science, University of the Balearic Islands, Cra. de Valldemossa, km 7,5, 07122 Palma (Illes Balears), Spain*

Received 1 July 2014; received in revised form 16 July 2015; accepted 17 July 2015

Available online 22 July 2015

Abstract

In this paper all uninorms locally internal in the region $A(e)$ (given by the complement in $[0, 1]^2$ of $[0, e]^2 \cup [e, 1]^2$, where e is the neutral element of the uninorm) having continuous underlying operators are studied and characterized, by distinguishing some cases. When the underlying t-norm and t-conorm are not given by ordinal sums, it is proved that uninorms locally internal in $A(e)$ are in fact all possible uninorms with these underlying operators (except when both the t-norm and the t-conorm are strict in which case there is also the class of representable uninorms), leading to a finite number of possibilities. When at least one of the continuous underlying operators is given by an ordinal sum, again there are other possible uninorms than those that are locally internal in $A(e)$, but all uninorms with this property are also characterized. In this case, infinitely many possibilities can appear depending on the set of idempotent elements of the uninorm.

Keywords: Uninorm; Locally internal operator; t-Conorm; t-Norm; Continuity

1. Introduction

Uninorms are a special kind of aggregation functions that generalize both t-norms and t-conorms [2,3,18,25]. Uninorms are increasing, commutative and associative binary operators on the unit interval having a neutral element $e \in [0, 1]$. They appear for the first time using the term uninorm in [30] (although the very related operators called *Dombi's operators* were already studied in [8]) with the idea of allowing certain kind of aggregation operators combining the maximum and the minimum, depending on an element $e \in]0, 1[$. This idea was deeper studied in [17], where the structure of such operators was analyzed and two first classes of uninorms were introduced: uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$, and representable uninorms (extremely related with Dombi's operators introduced in [8], see also [9]).

* Corresponding author. Fax: +34 971173003.
E-mail addresses: paweldrs@ur.edu.pl (P. Drygaś), daniel.ruiz@uib.es (D. Ruiz-Aguilera), jts224@uib.es (J. Torrens).

http://dx.doi.org/10.1016/j.fss.2015.07.015

From then, many papers have been published dealing with uninorms, specially from the point of view of their applications. Thus, uninorms have proved to be useful in a wide range of fields like aggregation of information, expert systems, neural networks, fuzzy system modeling, pseudo-analysis and measure theory, fuzzy mathematical morphology, fuzzy logic, approximate reasoning, and so on. Due to this great quantity of applications, uninorms have also been studied from the purely theoretical point of view. In this way, the following research lines have been investigated among others:

- Study of additional properties of uninorms, that usually derive in the solution of functional equations involving this kind of operators. Examples of this research line are the study of the properties of idempotency, modularity, migrativity, distributivity of uninorms between them and with other types of operators, among others.
- Study of some generalizations of uninorms. In this line we can highlight the introduction of the so-called *weak uninorms*, *left and right-uninorms*, *n-uninorms*, analysing their respective properties.
- Study of fuzzy implication functions derived from uninorms. In this line, using the fact that uninorms must be conjunctive or disjunctive, they have been used to define implication functions in the most usual ways, that is, RU-implications, (U, N)-, and QL and D-implications.

Moreover, additionally to the previous lines of investigation, another interesting one deals with the internal *structure of uninorms*, and it is in this line where this paper can be considered. The first step in the study of the structure of uninorms was done in [17], where the general structure of any uninorm as a special combination of a t-norm and a t-conorm was done. There, it is proved that any uninorm U with neutral element $e \in]0, 1[$ works as a t-norm T in $[0, e]^2$, as a t-conorm S in $[e, 1]^2$, and its values are between minimum and maximum in the set of points

$$A(e) = [0, e[\times]e, 1] \cup]e, 1] \times [0, e[.$$

In [17] classes of $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$, and representable uninorms were introduced. After this, some other classes of uninorms were introduced and characterized like idempotent uninorms in [6,23,29] and uninorms continuous in the open square $]0, 1[^2$ in [19] and [12]. Even the proper representable uninorms were more accurately characterized as those uninorms that are continuous in $[0, 1]^2 \setminus \{(1, 0), (0, 1)\}$ (see [27]), and also as those that are strictly increasing in the open square $]0, 1[^2$ with continuous underlying operators (see [15]).

There still exists another class of uninorms that has been extensively studied in the literature, i.e., those uninorms that are *locally internal* in the region $A(e)$. The term locally internal is used to represent the fact that the values of a uninorm U in any point $(x, y) \in [0, 1]^2$ must satisfy the property $U(x, y) \in \{x, y\}$. This property was studied first for monotonic and associative operators in general in [5], where it is proved the existence of a decreasing function g dividing the region where the operator is given by the minimum and by the maximum. These results were used later to characterize the idempotent uninorms (see [23] and the correction in [29]), and also for the study of the mentioned uninorms that are locally internal at least in the region $A(e)$ (see [10–14]). Unfortunately, although all these papers, a complete characterization of this class of uninorms is still unavailable, mainly because of the great quantity of possibilities for them.

On the other hand, another class of uninorms that has not been completely characterized until now is the class of uninorms with continuous underlying operators T and S. The first step in this direction was presented in the 2003 Linz seminar in [16] (see also [7] and [28] for the complete characterization in the discrete case, i.e., for uninorms defined on a finite chain). From then many efforts have been dedicated to this goal, but only some particular cases have been characterized. For instance, the case when both T and S are strict was done in [15] and revisited in [21], the case when both are nilpotent can be found in [22,26], and the cases when one is strict and the other is nilpotent were characterized in [21]. Furthermore, the case when T is nilpotent and S is maximum was also characterized in [26]. No more cases have been appeared until now, and specially no one dealing with T and/or S given by ordinal sums.

In this paper we want to characterize all uninorms with continuous underlying T and S in some other cases. On one hand, we characterize the cases combining T Archimedean (strict or nilpotent) with the maximum, and the minimum with S Archimedean. Moreover, from a detailed study of those uninorms that locally internal in $A(e)$, we also present a characterization of all uninorms with this property and with continuous underlying T and S given by ordinal sums. The paper is organized as follows. In Section 2 we give some preliminaries and notation that will be used along the paper. In Section 3 we analyse how can be in general the uninorms locally internal in $A(e)$ having continuous underlying operators, and then some subsections are devoted to characterize them in all cases when the underlying