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Left (right) semi-uninorms and coimplications on a complete lattice ☆

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Abstract

In this paper, we firstly discuss the deresidual operations of left (right) semi-uninorms and show which properties they satisfy. Then, we investigate the left and right semi-uninorms induced by a coimplication and give some conditions such that the operations induced by a coimplication constitute left or right semi-uninorms. Finally, we demonstrate that the meet-semilattice of all disjunctive right (left) $\wedge$-distributive left (right) semi-uninorms is order-reversing isomorphic to the join-semilattice of all right $\vee$-distributive coimplications, which satisfy the neutrality principle.

Keywords: Fuzzy connective; Left (right) semi-uninorm; Coimplication; Neutrality principle; Order property

1. Introduction

Uninorms, introduced by Yager and Rybalov [30], and studied by Fodor et al. [13], are special aggregation operators that have been proven useful in many fields like fuzzy logic, expert systems, neural networks, aggregation, and fuzzy system modeling (see [14,25,28,29]). This kind of operation is an important generalization of both t-norms and t-conorms and a special combination of t-norms and t-conorms (see [13]). But, there are real-life situations when truth functions may not be associative or commutative (see [11,12]). By throwing away the commutativity from the axioms of uninorms, Mas et al. introduced the concepts of left and right uninorms on $[0, 1]$ in [17] and later on a finite chain in [18], Wang and Fang [26,27] studied the left and right uninorms on a complete lattice. By removing the associativity and commutativity from the axioms of uninorms, Liu [16] introduced the concept of semi-uninorms on a complete lattice and Su et al. [23] discussed the notion of left and right semi-uninorms on a complete lattice. On the other hand, it is well known that a uninorm (semi-uninorm, left and right uninorms) U can be conjunctive or disjunctive whenever $U(0, 1) = 0$ or 1, respectively. This fact allows us to use uninorms (semi-uninorm, left and right uninorms) in defining fuzzy implications and coimplications (see [6,16,19,20,26,27]).

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In this paper, based on [6,16,20,21,27], we further study left (right) semi-uninorms and coimplications on a complete lattice. The organization of this study is as follows. In Section 2, we recall the adjoint functor theorem and some necessary definitions and examples about the left (right) semi-uninorms on a complete lattice. In Section 3, we discuss the deresidual operations of left (right) semi-uninorms, show that the right (left) deresidual operation of a disjunctive right (left) $\wedge$-distributive left (right) semi-uninorm is a right $\vee$-distributive coimplication, which satisfies the neutrality principle, and the left (right) deresidual operation of a strict left (right)-disjunctive left (right) $\wedge$-distributive left (right) semi-uninorm is a right $\vee$-distributive coimplication, which satisfies the order property. In Section 4, we investigate the left and right semi-uninorms induced by a coimplication and give some conditions such that the residual operations induced by a coimplication constitute left or right semi-uninorms. In Section 5, we reveal the relationships between disjunctive right (left) $\wedge$-distributive left (right) semi-uninorms and right $\vee$-distributive coimplications, which satisfy the neutrality principle.

The knowledge about lattices required in this paper can be found in [3].

Throughout this paper, unless otherwise stated, L always represents any given complete lattice with maximal element 1 and minimal element 0; J stands for any index set.

2. Adjunctions and left (right) semi-uninorms

In this section, we firstly recall the notion of adjunction between ordered sets and the adjoint functor theorem.

An adjunction also called isotonic Galois connection between order sets M and N, denoted by $f \dashv g$, is a pair of maps $f: M \to N$ (left adjoint of g) and $g: N \to M$ (right adjoint of f) satisfying the condition

$$x \le g(y) \Leftrightarrow f(x) \le y \quad \forall x \in M, y \in N;$$

equivalently, $f \dashv g$ if f and g are isotonic and satisfy the adjoint inequalities

$$\text{(AD1) } x \le g\big(f(x)\big)\ \forall x \in M, \quad \text{(AD2) } f\big(g(y)\big) \le y\ \forall y \in N.$$

When $M = N = L$, if $f \dashv g$, then

$$0 \le g(y) \Rightarrow f(0) \le y, \quad f(x) \le 1 \Rightarrow x \le g(1) \quad \forall x, y \in L,$$

i.e., $f(0) = 0$ and $g(1) = 1$.

It will be useful to refer to the so-called adjoint functor theorem, for ordered categories.

Theorem 2.1. *(See Della Stella and Guido [9], Guido and Toto [15].) Let $(M, \le)$ and $(N, \le)$ be ordered sets. Then the following statements hold.*

(1) *If $f \dashv g$, then f preserves existing joins and g preserves existing meets. Moreover, one has*

$$f(x) = \bigwedge \{y \in N \mid x \le g(y)\} \quad \forall x \in M,$$

and f is the unique left adjoint of g;

$$g(y) = \bigvee \{x \in M \mid f(x) \le y\} \quad \forall y \in N,$$

and g is the unique right adjoint of f.

(2) *If $(M, \le)$ is a complete lattice and $f: M \to N$ preserves $\bigvee$, then the function*

$$g: N \to M, y \mapsto g(y) = \bigvee \{x \in M \mid f(x) \le y\}$$

is the unique right adjoint of f.

(3) *If $(N, \le)$ is a complete lattice and $g: N \to M$ preserves $\bigwedge$, then the function*

$$f: M \to N, x \mapsto f(x) = \bigwedge \{y \in N \mid x \le g(y)\}$$

is the unique left adjoint of g.