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An introduction to quantaloid-enriched categories

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Abstract

This survey paper, specifically targeted at a readership of fuzzy logicians and fuzzy set theorists, aims to provide a gentle introduction to the basic notions of quantaloid-enriched category theory. We discuss at length the definitions of quantaloid, quantaloid-enriched category, distributor and functor, always giving several examples that – hopefully – appeal to the intended readership. To indicate the strength of this general theory, we explain in considerable detail how (co)limits are dealt with, and particularly how the Yoneda embedding of a quantaloid-enriched category in its free (co)completion comes to be. Our insistence on quantaloid-enrichment (rather than quantale-enrichment) is duly explained by examples requiring a notion of "partial elements" (sheaves, partial metric spaces). A final section glosses over some further topics, providing ample references for the interested reader.

Keywords: Quantale; Quantaloid; Enriched category; Multi-valued logic; Partial metric space; *t*-norm; Fuzzy set

1. Historical background

Categories, functors and natural transformations were first defined, in 1945, by Eilenberg and MacLane [12] as "a technical background for the intuitive notion of naturality", providing "opportunities for the comparison of constructions [...] in different branches of mathematics". In that paper they develop the basic notions of what we now call *category theory*, including e.g. limits and colimits, and give examples in homological algebra and algebraic topology. About a decade later, A. Grothendieck published his Tôhoku paper [17] on homological algebra (but paving the way for algebraic geometry too). Particularly his definition of *Abelian category* shows how a category is not merely a convenient tool to speak about a collection of mathematical structures, but is in fact a versatile mathematical structure in itself. That is to say, we explicitly have here *categories as structures*, as opposed to *categories of structures*.

In the Sixties Bénabou [3] made an abstraction of the notion of tensor product, defining *catégories avec multiplication* (monoidal categories); Eilenberg and Kelly [11] rather formalized the "internal hom" of a category, speaking of *closed categories*. Both Bénabou [3] and Eilenberg and Kelly [11] showed how such a monoidal/closed category $\mathcal{V}$ can serve as the base for $\mathcal{V}$-*enriched categories*. But it was Lawvere's paper [33], from 1973, with its deep insights (enriched presheaves, Cauchy completion) and convincing examples (posets, metric spaces), that made $\mathcal{V}$-enriched categories part of the working mathematician's toolbox. Kelly's book [30] became the standard reference on the subject.

Also in 1973, D. Higgs [24] showed how the topos $\mathsf{Sh}(L)$ of sheaves on a locale L can be described equivalently as a category of "L-valued sets", i.e. sets equipped with an equality relation taking truth values in L, thus exhibiting its multi-valued intuitionistic logic. However, such an L-valued set is easily seen *not* to be an L-enriched category—indeed,

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a slightly more general notion is due. In fact, back in 1967, Bénabou [5] had already defined *bicategories* and recognized that every such bicategory $\mathcal{W}$ can serve equally well as base for $\mathcal{W}$-enriched categories (which he called *polyads in* $\mathcal{W}$, for their kinship with *monads*). Walters [57] then showed how any locale L gives rise to a bicategory $\mathcal{R}(L)$ in such a way that sheaves on L can be described as (particular) $\mathcal{R}(L)$-enriched categories; and in [58], he generalized his argument to sheaves on a site. This encouraged Street [45–47] to further develop the theory and applications of $\mathcal{W}$-enriched categories, often also with coauthors [8,10].

This survey paper, written for a readership of fuzzy logicians and fuzzy set theorists, will be concerned with a particular instance of categories enriched in a bicategory, namely, where the base bicategory is a so-called *quantaloid* $\mathcal{Q}$: essentially, the local structure in the bicategory is posetal. This 'simplification' of the general theory still includes many important examples, such as ordered sets, (partial) metric spaces, sheaves, multi-valued logic and fuzzy sets; but it luckily does away with many a cumbersome "compatibility issue" so typical of bicategorical computations. A *quantale* is a one-object quantaloid (in other words, it is a monoidal closed poset), so the theory of quantaloid-enriched categories comprises the 'even simpler' theory of quantale-enriched categories—and indeed, the latter is already sufficiently general to include interesting examples such as ordered sets and metric spaces. However, we do insist on the use of quantaloid-enrichment: for one thing, quantaloids arise naturally as universal constructions, even when starting from a quantale; and for another, it is precisely with quantaloid-enrichment that we can elegantly express the notion of "partially/locally defined elements" (as well illustrated by the formulation of localic sheaves and of partial metric spaces as enriched categories).

Our modest ambition is only to explain and illustrate the basic definitions and a few emblematic results in quantaloid-enriched category theory, and we claim little or no originality for the mathematics contained in this paper. All of the concepts and results being by now quite standard notions in (enriched) category theory, we have not systematically traced their historical origins (apart from this short introduction). We do hope that the interested reader will find his or her way to the substantial literature on the subject.

2. Quantaloid-enriched categories

An ordered set $(X, \leq)$ can be thought of as a set X together with a binary predicate

$$X(-,-): X \times X \longrightarrow \{0,1\}: (y,x) \mapsto X(y,x) := \begin{cases} 1 & \text{if } y \leq x \\ 0 & \text{otherwise} \end{cases}$$

satisfying, for all $x, y, z \in X$,

$$X(z,y) \wedge X(y,x) \leq X(z,x) \quad \text{and} \quad 1 \leq X(x,x)$$

for transitivity and reflexivity.[1] The latter two conditions are equations in the Boolean algebra $\mathbf{2} = \{0,1\}$ which only make use of its order structure, its intersection and its top element. Thus we can repeat this predicative definition of ordered set over any set of "truth values" $T = (T, \leq, \circ, 1)$ which is ordered and comes with a multiplication and neutral element. For convenience we first make some extra assumptions on the set of "truth values".

Definition 2.1. A *quantale*[2] $Q = (Q, \bigvee, \cdot, 1)$ is a monoid $(Q, \cdot, 1)$ combined with a sup-lattice[3] $(Q, \bigvee)$ in such a way that, for all $f, g, (f_i)_i, (g_j)_j \in Q$,

$$g \cdot \left(\bigvee_i f_i \right) = \bigvee_i (g \cdot f_i) \quad \text{and} \quad \left(\bigvee_j g_j \right) \cdot f = \bigvee_j (g_j \cdot f).$$

A *homomorphism* $h: Q \longrightarrow Q'$ between quantales is a monoid homomorphism which preserves suprema.

[1] What we call an 'order' is often called a 'preorder'. For a transitive, reflexive and anti-symmetric relation we shall use the term 'partial order' or 'anti-symmetric order'.

[2] Mulvey [38] coined the term *quantale* as contraction of "quantum locale"; the study of locales as monoidal sup-lattices was initiated by [29].

[3] A sup-lattice $L = (L, \bigvee)$ is a partial order $(L, \leq)$ in which every $S \subseteq L$ has a supremum $\bigvee S$. This notion is obviously equivalent to that of a complete lattice, but with a bias toward the supremum as primitive ingredient. To wit, a sup-morphism between two sup-lattices is a map that preserves all suprema (and therefore also the order) but not necessarily the infima.