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Fuzzy terms

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Dedicated to Werner Gähler on the occasion of his 80th birthday.

Abstract

In this paper we will show how purely categorical constructions of terms are advantageous when investigating situations concerning uncertainty; more specifically where uncertainty comes from and how uncertainty is integrated when dealing with terms over selected signatures. There are basically two ways of invoking uncertainty for terms. On one hand, we may proceed by building composed monads where uncertainty is provided by some suitable monad composed with the traditional term monad. On the other hand, we can provide a strictly formal basis for term monads being created over categories themselves carrying uncertainty. This is the distinction between 'computing with fuzzy' and 'fuzzy computing' and the fundamental question raised by these constructions is where uncertainty resides in language constructions for logic. This paper also shows how the notion of signature often needs to be expanded to levels of signatures, in particular when dealing with type constructors. Such levels allow us to strictly delineate, e.g., primitive operations, type terms, and value level terms. Levels of signature will in this paper be exemplified by the construction of the signature of simply typed lambda calculus.
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1. Introduction

In foundations, we need to be careful about the distinction between 'logic for mathematics' and 'mathematics for logic', and we need to be respectful about not mixing metalanguages with object languages. In this paper we do 'mathematics for logic', or to be more precise, 'category theory for logic'. Category theory then is our metalanguage for creating sets of terms and term functors. We use Zermelo–Fraenkel set theory including the Axiom of Choice (ZFC) as part of the set-theoretic metalanguage for category theory.

In our view, terms are really the foundational cornerstones of logic, and we also have to be precise about the underlying signatures containing sorts and operators. Substitutions are morphisms in the Kleisli category for the particular term monad and therefore fuzziness in terms carry over canonically to fuzziness in substitutions.

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Type constructors manipulate sorts, and create new sorts, and this presents a situation where we need to understand the notion of levels of signatures. The concept of λ-terms can be handled within this realm, and this in fact provides the underlying notions of terms when moving towards fuzzy λ-calculus. Fuzzy description logic comes as a special case in these considerations.

The distinction between one-sortedness and many-sortedness is important as one-sorted constructions do not *per se* carry over to many-sorted situations. Here is also where type constructors need special attention, where underlying categories are important carriers of uncertainty.

Fuzziness is traditionally seen mostly as a level of truth, comparable with and as distinguished from probability. This provides a rather narrow set theoretic view of fuzzy logic as terms and sentences are left as crisp. In this paper we therefore strongly argue in favor of allowing uncertainty not just to be considered for algebras of truth values but also in relation to uncertainty values of operators. This means we open up the avenue for considering fuzzy terms, and thereby influencing fuzziness in sentences, and so on throughout all constructions of logic languages.

The history of expressions as appearing in argumentation and logic is manyfold. Further, logic in a historical perspective can be viewed in terms of the distinction between argument and argumentation. Focusing on argument invites to thinking that there is an overall logic, i.e., one language common to everybody in dialogues involving arguments. Focusing on argumentation leads to thinking about individual logics, i.e., communication between people is really communication between the logics adopted by respective individual. Individualization then also calls for making a distinction between the observer and the observation, i.e., the function and the function value.

Function symbols and abstractions have been dealt with ever since the late 19th century. Lambda-like expressions were proposed already by Frege [1,2] and Peano [3], and Hilbert's lectures followed by *Grundlagen* is a culmination in its own right. Contemporary developments of set theory are today the standard metalanguage, e.g., for category theory. During the time of Hilbert's lectures, Schönfinkel went a step further with his untyped combinators, although his ideas emerged before 1920, the paper, *Bausteine*, was not published until 1924. During the 1930s, Curry and Church worked intensively on developing groundwork for type theory and λ-calculus, and by 1940, simply typed λ-calculus had matured to its final form. Gödel and his incompleteness results are not directly related to the issue of terms, but rather to sentences and provability of sentences. In these respects, 'provability of sentence' is itself considered as a sentence, and this self-referentiality has indeed been under debate for almost a century. Whereas self-referentiality related to sets are seen leading to 'paradoxes', e.g., as pointed out by Russell, self-referentiality related to sentences leads to 'incompleteness'. Kleene develops deep results, e.g., for recursion culminating in his *Metamathematics*, but Kleene was surprisingly invisible during the course of further developments of type theory. The 1960s then is the time for model theory and forcing, and by the 1970s, category theory and universal algebra are already well established.

The outline of this paper is as follows. Sections 2–4 introduce not only the objectives and backgrounds but also provide the notation used in the subsequent formal constructions, more specifically, the necessary algebraic and categorical notions are briefly presented. In Section 5 we introduce many-sorted signatures as well as their fuzzy enrichment. Section 6 then covers the construction of term functors and monads over the defined signatures. In Section 7 we consider the application of such term monads in, e.g., type theory, λ-calculus, and description logic. Section 8 includes some notes on foundational matters and, finally, Section 9 concludes the paper and provides some perspectives for future investigations.

2. The need for formal descriptions of the term sets

In defining terms and term sets, the historical and contemporary sources in computing science and mathematics almost universally rely on 'verbal' definitions. For example, in [4] we have a definition similar to the following:

> Let $\Sigma = (S, \Omega)$ be a many-sorted signature with Ω as set of operators and S a set of sorts. Assume for each $s \in S$ a set of variables X_s. We then form terms as follows: A variable $x \in X_s$ is a term of sort s, constants $\omega :\to s$ are terms of sort s, and then inductively, if $\omega : s_1 \times \cdots \times s_n \to s$ is an n-ary operator, and $t_1, \ldots, t_n$ are terms of sort $s_1, \ldots, s_n$, respectively, then $\omega(t_1, \ldots, t_n)$ is a term of sort s.

This is then followed by a seemingly strict definitions of algebras:

> A Σ-algebra A for a signature $\Sigma = (S, \Omega)$ consists of a carrier set $A(s)$, for each $s \in S$. Further, for each $\omega : \longrightarrow s \in \Omega$, there is an element $A(\omega) \in A(s)$, for each $\omega : s_1 \times \cdots \times s_n \longrightarrow s \in \Omega$ with $n \geq 1$, there is a mapping
>
> $$A(\omega : s_1 \times \cdots \times s_n \longrightarrow s) : A(s_1) \times \cdots \times A(s_n) \longrightarrow A(s).$$

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