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Fuzzy sets and formal logics

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Dedicated to the memory of Franco Montagna

Abstract

The paper discusses the relationship between fuzzy sets and formal logics as well as the influences fuzzy set theory had on the development of particular formal logics. Our focus is on the historical side of these developments.

Keywords: Mathematical fuzzy logics; Fuzzy sets; Graded membership; Graded entailment; Monoidal logic; Basic fuzzy logic; Monoidal fuzzy logic; R-implications

1. Introduction

The theory of standard, i.e. crisp, sets is strongly tied with classical logic. This becomes particularly obvious if one looks at the usual set algebraic operations like intersection and union. These can for crisp sets A, B be characterized by the conditions

$$x \in A \cap B \quad \Leftrightarrow \quad x \in A \wedge x \in B\,, \tag{1}$$

$$x \in A \cup B \quad \Leftrightarrow \quad x \in A \vee x \in B\,. \tag{2}$$

The theory of fuzzy sets, as initiated in 1965 by Lotfi A. Zadeh [114], started with quite similar definitions for the membership degrees of the set algebraic operations:

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}\,, \tag{3}$$

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}\,, \tag{4}$$

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but offered also other operations for fuzzy sets, called "algebraic" by Zadeh, as, e.g., an algebraic product AB and an algebraic sum $A + B$ defined via the equations[1]

$$\mu_{AB}(x) = \mu_A(x) \cdot \mu_B(x)\,, \tag{5}$$

$$\mu_{A+B}(x) = \min\{\mu_A(x) + \mu_B(x), 1\}\,. \tag{6}$$

Zadeh [114] designed the fuzzy sets as a mathematical tool for the modeling of vague notions. Essentially he did not relate his fuzzy sets to non-classical logics. There was only a minor exception. In discussing the meaning of the membership degrees he explained (in a "comment" pp. 341–342, and with reference to the monograph [92] and Kleene's three valued logic) with respect to two thresholds $0 < \beta < \alpha < 1$ that one may

> "say that (1) "x belongs to A" if $\mu_A(x) \geq \alpha$; (2) "x does not belong to A" if $\mu_A(x) \leq \beta$; and (3) "x has an indeterminate status relative to A" if $\beta < \mu_A(x) < \alpha$".[2]

The overwhelming majority of fuzzy set papers that followed [114] also treated fuzzy sets in the standard mathematical context, i.e. with an implicit reference to a naively understood classical logic as argumentation structure.

Here we sketch the way fuzzy sets and the idea of membership grading have been strongly related to non-classical, particularly many-valued logics.

This has not been an obvious development. Even philosophically oriented predecessors of Zadeh in the discussion of vague notions, like Max Black in 1937 [4] and Carl Hempel in 1939 [82], did refer only to classical logic, even in those parts of these papers in which they discuss the problem of some incompatibilities of the naively correct use of vague notions with principles of classical logic, e.g., concerning the treatment of negation-like statements. And also the most direct forerunner of fuzzy set theory, Karl Menger, used in 1951 only classical logic [96].

This paper is structured as follows. After this introduction, we start from the early relationship between fuzzy sets and Łukasiewicz logic up to the introduction of t-norms into fuzzy set theory. Then we discuss t-norm based logics for fuzzy set theory, their semantics and their axiomatizations. These ideas give rise to a whole zoo of related formal logics, as well as to the idea of logics with graded notions of inference. Finally, as a matter of example to show the flexiblility of these logics, we take a look at two more application oriented formalisms: fuzzy logics for reasoning about probabilities, and a formal-logical treatment of (some of) Zadeh's basic ideas for approximate reasoning.

2. Fuzzy sets and Łukasiewicz logic

In parallel, and independent of the approach by Zadeh, the German mathematician Dieter Klaua presented in 1965/1966 two versions [88,89] for a cumulative hierarchy of so-called *many-valued* sets.[3] These many-valued sets had the fuzzy sets of Zadeh as a particular case.

Historically, Zadeh's approach proved to be much more influential than that of Klaua.

In Klaua's two versions [88,89] for a cumulative hierarchy of fuzzy sets he considered as membership degrees the real unit interval $\mathcal{W}_\infty = [0, 1]$ or a finite, m-element set $\mathcal{W}_m = \{\frac{k}{m-1} \mid 0 \leq k < m\}$ of equidistant points of [0, 1]. He also started his cumulative hierarchies from sets U of urelements. The infinite-valued case with membership degree set $\mathcal{W}_\infty = [0, 1]$ gives, in both cases, on the first level of these hierarchies just the fuzzy sets over the universe of discourse U in the sense of Zadeh.

Furthermore Klaua understood the membership degrees as the truth degrees of the corresponding Łukasiewicz systems L_∞ or L_m, respectively.

[1] The reader should be aware that equation (6) is not Zadeh's original formulation. He introduced the algebraic sum as a partial operation for fuzzy sets, defined only if $\mu_A(x) + \mu_B(x) \leq 1$ was always satisfied. We disregard this minor(?) difference here.

[2] The cautious reader should be aware that we use here the more common notation μ_A instead of f_A from [114].

[3] The German language name for these objects was "mehrwertige Mengen". The stimulus for these investigations came from discussions following a colloquium talk which K. Menger had given in Berlin (East) in the first half of the 1960s, cf. [91].