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Integrals based on monotone set functions

Erich Peter Klement [a,b,*], Jun Li [c], Radko Mesiar [d,e], Endre Pap [f,g]

[a] *Johannes Kepler University, 4040 Linz, Austria*
[b] *JKU Softwarepark Hagenberg, 4232 Hagenberg, Austria*
[c] *School of Sciences, Communication University of China, Beijing 100024, China*
[d] *Faculty of Civil Engineering, Slovak University of Technology, 813 68 Bratislava, Slovakia*
[e] *University of Ostrava, 701 03 Ostrava, Czech Republic*
[f] *Singidunum University, 11000 Belgrade, Serbia*
[g] *Óbuda University, 1034 Budapest, Hungary*

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Abstract

An overview of various integrals is given which can be defined on arbitrary monotone set functions vanishing in the empty set (called here monotone measures). Our survey includes not only the Choquet integral (1954) [10], the Shilkret integral (1971) [66] and the Sugeno integral (1974) [71] and some of their properties, but also some more general and more recent concepts as universal integrals Klement et al. (2010) [27] and decomposition integrals Even (2014) [13], together with some of their properties, such as integral inequalities and convergence theorems.

Keywords: Monotone measure; Choquet integral; Shilkret integral; Sugeno integral; Universal integral; Decomposition integral

1. Introduction

The earliest traces of computations which were called much later *integration* are related to the problem of finding the volume, the length, the area or the surface of a geometrical object. Constructive approaches to "integration" can be traced back to ancient Egypt around 1850 BC: the *Moscow Mathematical Papyrus* (Problem 14) contains a formula for the frustum of a square pyramid. The first documented systematic technique to determine "integrals" is the method of exhaustion of the Greek astronomer Eudoxus (ca. 370 BC) who suggested to find areas and volumes by approximating them by objects whose area or volume was known. This method was further explored in ancient Greece by Archimedes (third century BC) and, independently, in China (Liu Hui in the third century AD, Zu Geng around 500 AD) and in India (Aryabhata, also around 500 AD).

* Corresponding author at: Johannes Kepler University, 4040, Linz, Austria. Tel.: +43 732 2468 4141; fax: +43 732 2468 4142.
E-mail addresses: ep.klement@jku.at (E.P. Klement), lijun@cuc.edu.cn (J. Li), mesiar@math.sk (R. Mesiar), pape@eunet.rs (E. Pap).

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In all these problems additivity was an intrinsic property, eventually leading to the classical *Riemann integration*. This method was introduced by G.W. Leibniz [34] and I. Newton [49] in the 17th century, further developed by A.L. Cauchy [8] in the beginning of the 19th century, and formalized by B. Riemann [60] in the middle of the 19th century. This whole development was done without using the notion of measure (only the additivity and the volume of a cuboid $[x_1, y_1] \times \cdots \times [x_n, y_n]$ given by $(y_1 - x_1) \cdots (y_n - x_n)$ were needed — the underlying standard Lebesgue measure on the Borel subsets of $\mathbb{R}^n$ was somewhat hidden). As an extension and completion of additive integration, the concept of a nonnegative σ-additive measure and the corresponding *Lebesgue integral* were introduced in 1902 by H. Lebesgue [31].

Nevertheless, non-additive concepts were already proposed several years before, e.g., by the economist C. Menger [38] in the context of interacting criteria. Later on, several not necessarily additive set functions were considered and studied in detail (see [54,74] for a bibliography, and also [35]).

To the best of our knowledge, the first trace of an integral based on a non-necessarily additive set function can be found in 1925 in the work of G. Vitali [73]. Vitali's integration based on outer and inner measures is a predecessor of the *Choquet integral* introduced by G. Choquet in 1954 [10] for special monotone set functions called *capacities*. Note that Choquet has considered several kinds of set functions, all of which being monotone and vanishing for the empty set.

Another approach to integration without additivity was given by N. Shilkret in 1971 [66] where *maxitive measures* were considered (note, however, that the Shilkret integral can be defined for each monotone measure).

Yet another concept was proposed by M. Sugeno in 1972 (English publication in his PhD Thesis in 1974 [71]), where *fuzzy measures* were used to deal with suitable "expectations" of fuzzy events (the *probabilities of fuzzy events* considered in 1968 by L.A. Zadeh [77] were standard probabilistic expectations).

In the following years, the properties of these integrals were studied in detail, and some new types of integrals were proposed. Much of this work was done by colleagues working in decision theory, artificial intelligence, etc., but also by members of the "fuzzy set community". An early trace is [69], but also some (unpublished) work by U. Höhle in this context goes back to the early eighties (for papers which were published later and contain some of this work see [20–22]). Many important results can be found in the edited volume [19], in the overview chapter [6] and in [61–63]. In [11, p. viii] D. Denneberg describes the situation as it was in 1994 (and to a large extent still is today):

> "Altogether one can observe that the fundamentals of non-additive integration had been developed independently again and again. There is no unified, widely known and accepted approach to the theory."

In this survey, we will discuss only integrals which are defined for non-negative real functions (which, if their range is contained in $[0, 1]$, can be viewed as membership functions of fuzzy events). Concerning the underlying measures, we restrict ourselves to real-valued monotone measures. Throughout the paper, we consistently will speak about monotone measures, even if the original sources have used an alternative terminology (such as capacities or fuzzy measures). If we consider monotone measures with additional properties, we shall either use standard names (such as probability measure) or we will explicitly mention the additional properties (e.g., supermodular monotone measure).

In the following section, we give some preliminaries and present the three basic integrals based on monotone measures, namely, the Choquet [10], the Shilkret [66], and the Sugeno integral [71]. In Section 3 a deeper discussion concerning the Choquet integral is given. Section 4 deals in more detail with the Shilkret and the Sugeno integral, and with some related integrals. In both sections some convergence theorems and some integral inequalities are presented. Section 5 is devoted to universal integrals [27], a unified approach covering the three basic monotone measure-based integrals. Here we recall some special classes of integrals, most of them in connection with copulas [68]. In Section 6 the decomposition-integral [13] and related integrals are discussed. Finally, we briefly mention also some other types of integrals, such as integrals based on pseudoadditive measures or bipolar integrals.

2. Basic monotone measure-based integrals

We start with a fixed non-empty set X, equipped with a σ-algebra $\mathcal{A} \subseteq 2^X$ of subsets of X. The pair $(X, \mathcal{A})$ will be called a *measurable space*. Note that in some of the cases discussed later on, coarser classes of subsets of X can be considered (the interested reader can find them in the original sources). If the set X is finite we shall always