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Convergence theorems for monotone measures

Jun Li [a], Radko Mesiar [b,c], Endre Pap [d,e], Erich Peter Klement [f,g,*]

[a] *School of Sciences, Communication University of China, Beijing 100024, China*
[b] *Faculty of Civil Engineering, Slovak University of Technology, 813 68 Bratislava, Slovakia*
[c] *University of Ostrava, 70103 Ostrava, Czech Republic*
[d] *Singidunum University, 11000 Belgrade, Serbia*
[e] *Óbuda University, 1034 Budapest, Hungary*
[f] *Johannes Kepler University, 4040 Linz, Austria*
[g] *JKU Softwarepark Hagenberg, 4232 Hagenberg, Austria*

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Abstract

In classical measure theory there are a number of convergence theorems, such as the Egorov, the Riesz and the Lusin theorem, among others. We consider monotone measures (i.e., monotone set functions vanishing in the empty set and defined on a measurable space) and discuss, how and to which extent classical convergence theorems can be carried over to this more general case.

Keywords: Monotone measure; Monotone set function; Convergence theorems; Egorov theorem; Riesz theorem; Lusin theorem

1. Introduction

The term “measure” essentially goes back to H. Lebesgue [33] in the beginning of the 20th century who understood it in the sense of a σ*-additive measure* and used it in order to generalize the classical Riemann integral to what is known now as the Lebesgue integral. The σ-additivity is a powerful property (implying, among others, continuity and the vanishing in the empty set), and it is particularly useful when considering convergence (also in probability theory). A short history of measure theory can be found, e.g., in [76].

Even before Lebesgue “measures” with weaker types of additivity were considered: a prominent example is the *content*, i.e., a finitely additive measure [85] (also called charge in [3]).

There are also some concepts of measures which are not additive in a narrow sense, but only related to additivity. Here we can mention outer measures [4], superadditive and subadditive measures [92], modified probabilities [54], capacities [5], maxitive measures [82], upper and lower probabilities [81], belief and plausibility measures [6,81], sub- and supermodular measures [5], null-additive measures [73], and k-additive measures [13,55].

* Corresponding author at: Johannes Kepler University, 4040 Linz, Austria.
E-mail addresses: lijun@cuc.edu.cn (J. Li), mesiar@math.sk (R. Mesiar), pape@eunet.rs (E. Pap), ep.klement@jku.at (E.P. Klement).

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In his Ph.D. thesis, M. Sugeno [86] went a step farther: for what he called a *fuzzy measure* he did not require any property related to the addition, but only the vanishing in the empty set and the monotonicity with respect to set inclusion (and some continuity).

These non-additive measures (and the corresponding integrals) in the sense of M. Sugeno [86] have been widely studied during the past 40 years, and many results have been obtained, see [7,15,28,29,31,32,34,56,59,60,73–75,79, 91,96].

As a matter of fact, these so-called fuzzy measures do not involve any fuzziness (as introduced in [103]). Various other names have been used for them (and slight variations thereof) over the years: monotone measures, non-additive measures, capacities, etc. (see, e.g., [39,40,57,58,62,96]). But the essential feature, namely, for nonnegative set functions to drop any form of additivity and to replace it by the (usually weaker) monotonicity (and the vanishing in the empty set) was a significant step toward a generalization of the theory of measures.

In this survey, our basic concept will be that of real-valued, nonnegative monotone measures. Throughout the paper, we consistently will use the term *monotone measure* for a real-valued, nonnegative set function defined on a certain class of classical sets which vanishes in the empty set and which is monotone with respect to set inclusion. In doing so, on the one hand we emphasize the monotonicity as the key property of these set functions, and on the other hand we shall avoid any confusion with similar notions like capacities, monotone set functions, non-additive measures or fuzzy measures (even if these notions were used in the original sources). If we consider monotone measures with additional properties, we shall either use standard names (such as probability measure) or we will explicitly mention the additional properties: the Lebesgue measure and each probability measure, for instance, are examples of σ-additive monotone measures.

In classical measure theory, i.e., for σ-additive monotone measures, there are several important convergence theorems, such the Egorov theorem [16,102], the close relationship between convergence almost everywhere and convergence in measure (sometimes called the F. Riesz–Lebesgue theorem [1]) and the Lusin theorem [53], etc. They describe implications between three concepts of convergence for sequences of real-valued measurable functions: almost everywhere convergence, almost uniform convergence, and convergence in measure ([16,102]).

If only the above-mentioned three concepts (convergence almost everywhere, almost uniform convergence, and convergence in measure) are considered in classical measure theory, there are six possible implication relations between them. Three of these relations are fully described by the theorems indicated above.

However, these classical theorems do not hold in general, when we move from σ-additive monotone measures to arbitrary monotone measures.

When considering arbitrary monotone measures, the lack of additivity always leads to new concepts of convergence (see Definition 3.2 below): for instance, we do not only have convergence almost everywhere but also convergence pseudo-almost everywhere, and so on. As a consequence, there are 30 implication relations to be studied.

Over the past 30 years a lot of effort has been put into studies to which extent the above-mentioned classical convergence theorems could be carried over to the more general case of monotone measures. Initially, most results dealt with the original concept of M. Sugeno [86] (see also [79]), i.e., with nonnegative set functions which are monotone, vanish at the empty set, and are continuous both from below and from above, and a lot of results were obtained in this context (see, among others, [18,47,51,73,79,87,93–95,99]).

Later on it turned out that the requirements of continuity from below and/or above could be fully dropped or substantially weakened, which led an improvement of many of the previous results. For instance, in the case of the classical Egorov and Riesz theorem and the statement that, for finite measures, convergence almost everywhere implies convergence in measure, necessary and sufficient conditions were given for these results to hold also for monotone measures [35–37,39,43,44,46,65,84,88,89].

In this survey, we first introduce and fix the basic preliminaries and notations for monotone measures and the different types of convergence of measurable functions in Sections 2 and 3. Then we consider the classical Egorov theorem (Section 4), the relationship between "convergence almost everywhere" and "convergence in measure" (Section 5), the classical Lusin theorem (Section 6), and the relationship between the concepts "fundamental in measure" and "convergence in measure" (Section 7), and we present the versions of these results obtained when moving from classical, i.e., σ-additive monotone measures, to arbitrary monotone measures. In each of these sections, a brief overview of the historical development is given. The proofs of the results mentioned here are generally omitted for the sake of brevity and readability — the interested reader is referred to the original sources which are always given. Finally, we only mention some further extensions when replacing the real line by some more general concepts.