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Fuzzy sets as two-sorted algebras

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Abstract

This paper makes the attempt to explain the foundations of fuzzy sets from the point of view of universal algebra. The free fuzzy set and the fuzzy power set are constructed. Moreover, fuzzy power sets give rise to a monad. In this context, fuzzy power sets appear as free fuzzy modules generated by their underlying fuzzy sets.

Keywords: Frobenius ℓ-monoid; 2-Sorted algebra; Fuzzy set; Fuzzy power set; Fuzzy module

1. Introduction and motivation

The fiftieth anniversary of fuzzy sets is a good occasion to go back to their roots and to explain the impact of universal algebra on fuzzy set theory. Therefore I do not consider fuzzy sets as *generalized characteristic functions* (cf. [7]) or as *enriched presheaves* (cf. [9]), but simply as *two-sorted algebras* without any further specific meaning. Before I put down the axioms of this type of algebra it seems to be interesting to consult first the well known paper on fuzzy sets [18] published by L.A. Zadeh in 1965, even though the principal idea of a fuzzy set goes back to K. Menger (cf. [14]).

In Section 2 of [18] a fuzzy set (class) A in X is not defined, but characterized by a function $X \xrightarrow{f_A} [0,1]$ which L.A. Zadeh understands as *membership (characteristic) function* assigning a real number of the interval $[0, 1]$ to each point in X. Also S. Mac Lane in his critical review on *the health of mathematics* admits that this idea is attractive and gives the following explanation of this function (cf. p. 54 in [13]):

> … instead of saying that an element x is or is not in the set A, let us measure the likelihood that x is in A.

As I said before I do not continue the debate on membership and measurement of membership, but I understand simply a fuzzy set in X as map from X to $[0, 1]$. The interesting question is now which algebraic properties of $[0, 1]$ have been used by L.A. Zadeh for his calculus of fuzzy sets. A short glance at [18] shows that he has selected four operations

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– one unary operation sending each element $\alpha \in [0,1]$ to $1-\alpha$,
– three binary operations given by the minimum, maximum and product,

and a partially defined operation determined by the addition which he calls *algebraic sum* (cf. p. 344 in [18]). For the contemporary it is unclear why L.A. Zadeh did not complete the algebraic sum to a totally defined binary operation which would coincide with the t-conorm corresponding to *Łukasiewicz arithmetic conjunction* determined by:

$$\alpha * \beta = \max(\alpha + \beta - 1, 0), \qquad \alpha, \beta \in [0,1].$$

But in a more mathematical language it is clear that L.A. Zadeh has used the standard structure of an MV-algebra on $[0,1]$ (cf. [4,8])

$$([0,1], \min, \max, *)$$

in which the product $\cdot$ plays the role of a dominating binary operation (cf. [17]) — i.e.

$$(\alpha \cdot \beta) * (\gamma \cdot \delta) \leq (\alpha * \gamma) \cdot (\beta * \delta), \qquad \alpha, \beta, \gamma, \delta \in [0,1].$$

Since MV-algebras are divisible and commutative Frobenius ℓ-monoids (cf. [1]), I put down the following abstract definition of a fuzzy set:

A triple $(X, f, \mathfrak{Q})$ is called a *fuzzy set* iff X is a set, $\mathfrak{Q}$ is a Frobenius ℓ-monoid and $X \xrightarrow{f} \mathfrak{Q}$ is a map.

The aim of this paper is to show that fuzzy sets in the previous sense are two-sorted algebras — i.e. the underlying Frobenius ℓ-monoid is incorporated into the definition of a fuzzy set and the membership map f is viewed as a unary two-sorted operation (cf. [6,2]).[1] The precise definition of the corresponding two-sorted signature will be given in Section 4. Proceeding in this way the techniques of universal algebra are available, and it is not surprising that *free fuzzy sets* exist. In particular, every fuzzy set is a *quotient* of a free fuzzy set.

Moreover, in the case of Frobenius quantales the fuzzy power set construction and its monadic basis are explained. In this context, fuzzy power sets appear as free fuzzy modules generated by their underlying fuzzy sets.

As a preparation I first explain some basic properties of Frobenius ℓ-monoids.

2. Frobenius ℓ-monoids

A *residuated ℓ-semigroup* is a quintuple $(X, \leq, *, \searrow, \swarrow)$ where $(X, *)$ is a semigroup, $(X, \leq)$ is a lattice with the corresponding binary lattice operations $\wedge$ and $\vee$, and $\searrow$ and $\swarrow$ are two further binary operations on X satisfying the following condition for all $x, y, z \in X$ (cf. [1]):

$$y \leq x \searrow z \iff x * y \leq z \iff x \leq z \swarrow y. \tag{AD}$$

In this context $\searrow$ is called the *right-implication* and $\swarrow$ is called the *left-implication* w.r.t. the semigroup operation $*$.

The next proposition collects some fundamental properties of residuated ℓ-semigroups.

Proposition 2.1. *Let $((X, \leq), *, \searrow, \swarrow)$ be residuated ℓ-semigroup. Then the semigroup operation $*$ is distributive on binary joins in each variable separately — i.e.*

$$x * (y \vee z) = (x * y) \vee (x * z) \quad \text{and} \quad (x \vee y) * z = (x * z) \vee (y * z). \tag{D}$$

If $\perp$ is the universal lower bound of $(X, \leq)$, then $\perp$ is always the zero element of $(X, *)$. Moreover the following properties hold for all $x, y, z \in X$:

(i) $(x \vee y) \searrow z = (x \searrow z) \wedge (y \searrow z)$ *and* $z \swarrow (x \vee y) = (z \swarrow x) \wedge (z \swarrow y)$.
(ii) $x \searrow (y \wedge z) = (x \searrow y) \wedge (x \searrow z)$ *and* $(z \wedge y) \swarrow x = (z \swarrow x) \wedge (y \swarrow x)$.

[1] It is interesting to see that the term *many-sorted* has its origin in mathematical logic (cf. [16]) and plays now a prominent role in certain areas of algebra (cf. [15]).