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An overview of fuzzy logic connectives on the unit interval

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Abstract

The aim of this short paper is to give a simple look to the historical development of logical connectives for fuzzy sets and fuzzy logic. Concepts that have been (and still are) in the core of extensive theoretical research, like conjunction, disjunction, complement, subsethood and fuzzy conditionals, are considered in an informal way. An extensive list of essential references helps the interested reader to find sources for deeper study of the main subjects.

Keywords: Fuzzy connectives and aggregation operators; t-Norms; t-Conorms; Negations; Uninorms; Implication functions

1. Introduction

In his seminal work [71] on fuzzy sets, Lotfi A. Zadeh suggested operations min, max and $N_c(x) = 1 - x$ to functionally express the intersection, the union and the complement of fuzzy sets, respectively. Given fuzzy sets A, B on a universe X, with membership functions $\mu_A, \mu_B : X \to [0, 1]$, the intersection, the union and the complement are new fuzzy sets whose membership functions are respectively given by

$$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x)), \quad \mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x)), \text{ and } \quad \mu_{A^c}(x) = 1 - \mu_A(x), \quad x \in X.$$

The use of these operations in the management of fuzzy sets has been implemented in many real problems and they have proved to be useful not only in fuzzy control and approximate reasoning, but also in many applications from image processing or computing with words, data mining and information retrieval, optimization problems, decision making and so on.

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2. From min and max to uninorms and nullnorms via t-norms and t-conorms

Although min, max and N_c have proved to be "good" logical connectives for fuzzy sets, many other alternatives have been investigated, trying to deal with different problems having specific characteristics and requirements. Already in [71], Zadeh defined other operations on fuzzy sets, under the name *algebraic product* and *algebraic sum*. Using contemporary terminology, the first one is called the *product t-norm*, while the second one is the *Łukasiewicz t-conorm*.

The most usual and widely known extensions of these connectives are *t-norms*, *t-conorms* ([3] or [22]) and (strong) *fuzzy negations* [67]. Although t-norms and t-conorms have appeared earlier in the theory of probabilistic metric spaces (see the book [66] for a first collection of results in this framework), where they still have a high impact, they have experimented a special development from their irruption in fuzzy set theory.

Many of the known results on t-norms and t-conorms, as well as their relation with fuzzy sets and fuzzy logic, were collected in some books entirely devoted to this topic (one of the first books with some chapters on logical connectives for fuzzy sets was [27]). For instance, definitions and properties, algebraic aspects, additive generators, representation and classification theorems, construction methods and parametrized families of t-norms can be found in [39]. Similarly, [2] is another book on t-norms, complementary to the previous one, that focuses on functional equations involving t-norms and copulas. On the other hand, the edited volume [38] is divided into 16 different chapters that concentrate on new aspects on t-norms, functional equations involving them, extensions of t-norms to discrete and interval valued settings, t-norm based fuzzy logics, t-norms and fuzzy measures, left-continuous t-norms and so on. In all mentioned books, many open problems concerning t-norms were posed. Some of them already solved in very recent papers, but many others remain still unsolved. Moreover, new papers constantly appear solving some open question about t-norms and t-conorms, dealing with new properties (like for instance convex combinations of triangular norms, see [23,56,57], or α-migrativity, see [23,28,29]) and even new open problems appear like in [54].

With respect to complements, not only strong fuzzy negations characterized in [67], but also more general functions, even non continuous, have been used in fuzzy set theory. See for instance [13] for weak negations or [65] for negations symmetrical with respect to the identity. However, strong negations are the most used ones to manage duality properties between t-norms and t-conorms, De Morgan laws, and so on.

Two new families of aggregation operators, *uninorms* and *nullnorms*, appeared in the literature as common extensions of t-norms and t-conorms. These operators preserve commutativity and associativity properties. Uninorms allow the neutral element to be any point $e \in [0, 1]$ (for t-norms $e = 1$, while for t-conorms $e = 0$, see [70] and [30]), whereas nullnorms (also called t-operators, see [12] and [43]) can have an arbitrary absorbing element $k \in [0, 1]$ (for t-norms $k = 0$, and for t-conorms $k = 1$). Any nullnorm with absorbing element k is constantly equal to k in all points (x, y) with $\min(x, y) \leq k \leq \max(x, y)$. Notice that any uninorm U keep also the property to be either conjunctive, that is $U(1, 0) = 0$, or disjunctive, that is $U(1, 0) = 1$, and so they can be used as new logical connectives for modeling intersections and unions.

Since their introduction both uninorms and nullnorms have attracted considerable attention. It has been revealed that the subject is theoretically challenging, and useful in many applied fields. The results on investigating different aspects of these operators can be found in papers published in diverse journals. Much of these works are devoted to one or some of the following topics:

- Study and/or characterization of different families of uninorms, like uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ (see [30]), representable uninorms [30,15,63], idempotent uninorms [14,42,65], uninorms continuous in the open unit square [35], locally internal uninorms [16,18–20]. The full description of uninorms with continuous underlying operators is a deeply investigated problem from its first introduction [26], with some partial solutions [15,21,40,41,58].
- Study of some families of uninorms with specific properties or with non-continuous underlying operators [33,34].
- Analysis of additional properties for uninorms and nullnorms that usually lead to the solution of functional equations involving them. For instance, idempotency [14,42,65], distributivity [17,24,45,46,60,62], modularity [44], Frank and Alsina equations [12], absorption [11], migrativity [9,10,47,48,55], and so on.
- Construction of fuzzy implication functions from uninorms, a topic that will be discussed in the next section.