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A frequentist view on cycle-transitivity of reciprocal relations

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Abstract

We establish a connection between two transitivity frameworks: the transitivity of fuzzy relations based on a commutative quasi-copula and the cycle-transitivity of reciprocal relations w.r.t. the dual quasi-copula as an upper bound function. Loosely speaking, it turns out that the latter can be characterized by imposing a lower bound on the relative frequency with which the former is fulfilled, when applied to reciprocal relations. We provide two compelling cases: the $4/6$ theorem, expressing that the winning probability relation of a set of independent random variables is at least 66.66% product-transitive, and the $5/6$ theorem, expressing that the mutual rank probability relation associated with a given poset is at least 83.33% product-transitive. Moreover, these lower bounds turn out be rather conservative, illustrating that, from a frequentist point of view, transitivity is abundant.

Keywords: Cycle-transitivity; Quasi-copula; Mutual rank probability relation; Reciprocal relation; Winning probability relation; Transitivity

1. Introduction

Given the key role that the basic concept of a mathematical relation has played in the development of discrete mathematics and computer science in the last century, it is not surprising that it was one of the first concepts that was subjected to the process of fuzzification in the early years of fuzzy set theory [39]. Its (mainly) two-dimensional nature is ideally suited to model relationships among the objects belonging to a given class or between the objects belonging to two given classes. Obviously, the transition from a Boolean, crisp setting to a gradual setting called for a set of tools apt to manipulate this new concept. Given the unipolar nature of fuzzy relations, generalizations of logical connectives are ideally suited for this purpose. In this sense, the development of the calculus of fuzzy relations profited from the never-ending expansion of the knowledge on aggregation functions [30]. A well-known example is the standard use of a triangular norm T for defining the property of T-transitivity of a fuzzy relation R on a universe X as the set of inequalities

$$T(R(x,y), R(y,z)) \leq R(x,z)\,, \tag{1}$$

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to be fulfilled for any $x, y, z \in X$. This property appears in numerous studies on fuzzy order relations [7], similarity relations [21] and preference relations [22,23,26]. The use of a triangular norm T has become unquestioned, to the extent that researchers do not wonder whether associativity (e.g. to ensure the existence of a T-transitive closure [9]) is really needed or not. Once again, we will show in this paper that 1-Lipschitz continuity cannot be ignored either.

Obviously, contrary to the spirit of fuzzy set theory, the notion of T-transitivity is again a Boolean one. For that purpose, various researchers have developed gradual versions of T-transitivity [3,29], for instance expressing the degree of T-transitivity $\mathrm{tr}(R)$ of a fuzzy relation R on X as

$$\mathrm{tr}(R) = \inf_{(x,y,z)\in X^3} I_T(T(R(x,y), R(y,z)), R(x,z)),$$

where I_T is the residual implicator of T, or using more general expressions based on fuzzy inclusion degrees, inspired by the fact that T-transitivity of a fuzzy relation R can be expressed as the inclusion $R \circ_T R \subseteq R$ (where $\circ_T$ stands for the sup-T composition of fuzzy relations). This degree of T-transitivity is inspired by the following property of the residual implicator I_T of a left-continuous t-norm: for any $a, b \in [0,1]$ it holds that

$$a \le b \Leftrightarrow I_T(a,b) = 1.$$

Such an approach to gradual properties was recently perfectionized by Běhounek et al. [4]. Despite the fact that this degree of T-transitivity takes values in [0, 1], it is still extremely conservative, as a single failure, i.e. the existence of $x, y, z \in X$ such that $I_T(T(R(x,y), R(y,z)), R(x,z)) \le \alpha$ will entail $\mathrm{tr}(R) \le \alpha$.

In recent years, reciprocal relations, sometimes confusingly presented as *fuzzy* relations $Q : X^2 \to [0,1]$ that satisfy the reciprocity condition $Q(a,b) + Q(b,a) = 1$, have witnessed increased attention. Their use in fuzzy set theory can be traced back to the work of Blin and Whinston [5,6] and Kacprzyk et al. [33] on consensus in fuzzy decision making. Just as fuzzy relations can be seen as generalizations of crisp relations, reciprocal relations can be seen as generalizations of the 3-valued representation of complete crisp relations [27]. Obviously, this induced bipolar semantics renders the application of generalizations of logical connectives, and derived properties such as T-transitivity, rather meaningless, although this was not always realized at first [37,38]. However, other fields of study have been using reciprocal relations (often called stochastic relations referring to their origin) for decades and have developed appropriate generalizations of properties of complete crisp relations, in particular of transitivity. The exponent of this development is undoubtedly the cycle-transitivity framework developed by some of the present authors, among others [10,13]. However, despite the generality of this framework, it suffers from the same shortcoming as T-transitivity: cycle-transitivity is again a Boolean notion.

Reciprocal relations abundantly appear in probabilistic settings, for instance as the winning probability relation expressing the winning probabilities among the components of a random vector [11,25], or as the mutual rank probability relation expressing the probability of one object of a poset being ranked higher than another one in a randomly selected linear extension of this poset [12,25]. The latter finite settings have been scrutinized over the past years, leading to a panoply of insights into the transitivity of these relations, always framed in the cycle-transitivity framework.

The purpose of this paper is to share some remarkable findings. In a finite setting, instead of computing a degree of T-transitivity, one might simply opt to count the number of times inequality (1) is fulfilled, and compute the relative frequency thereof. What we will show is that there exists a wide range of types of cycle-transitivity of reciprocal relations that can be expressed equivalently by stating a lower bound on the mentioned relative frequency of a related type of transitivity that would rather be reserved for fuzzy relations. In particular, we will reveal two fascinating theorems:

(i) **The** $4/6$ **theorem**: the winning probability relation of any set of independent random variables is at least $4/6 \times 100\%$ $T_{\mathbf{P}}$-transitive;
(ii) **The** $5/6$ **theorem**: the mutual rank probability relation of any poset is at least $5/6 \times 100\%$ $T_{\mathbf{P}}$-transitive;

where $T_{\mathbf{P}}$ stands for the product t-norm, i.e. $T_{\mathbf{P}}(x,y) = xy$.

This paper is organized as follows. In Section 2, we consider a generalization of T-transitivity, using conjunctors, quasi-copulas and copulas, and investigate the consequences of an (inadvertent) application to reciprocal relations.