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Short communication

A note on the continuity of triangular norms ☆

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Abstract

It is shown that for an associative function on the unit interval with certain boundary conditions, separate continuity implies joint continuity. This answers a question on triangular norms raised by Alsina, Frank, and Schweizer in 2003.

Keywords: Triangular norms; Continuity

1. Introduction

Triangular norms (t-norms, for short) were introduced by Schweizer and Sklar [9] in the study of probabilistic metric spaces as a special kind of associative functions defined on the unit interval. These functions have found applications in many areas since then. In particular, (continuous) t-norms and (continuous) triangular conorms play a prominent role in fuzzy set theory [4,6].

The notion of t-norms has been extended to the general setting of bounded partially ordered sets [3]. Because of the importance of continuous t-norms in fuzzy logic, continuity of t-norms on bounded partially ordered sets, the unit square $[0, 1]^2$ in particular, has been discussed in [3,5].

This note presents an answer to a question on continuous t-norms on the unit interval raised by Aslina, Frank, and Schweizer in 2003.

2. The question

Definition 1. (See Alsina et al. [1,2].) A t-norm on [0, 1] is a function $T : [0, 1]^2 \longrightarrow [0, 1]$ such that for all x, y, z, w in [0, 1],

(a) $T(x, 0) = T(0, x) = 0$, $T(x, 1) = T(1, x) = x$,
(b) $T(x, y) = T(y, x)$,

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(c) $T(x, y) \leqslant T(z, w)$ whenever $x \leqslant z$, $y \leqslant w$,
(d) $T(T(x, y), z) = T(x, T(y, z))$.

Further, if T is continuous with respect to the standard topology on $[0, 1]$, then it is called a continuous t-norm.

It is clear that $T(x, 0) = T(0, x) = 0$ can be derived from (c) and the condition that $T(x, 1) = T(1, x) = x$. The following characterization of continuous t-norms due to Mostert and Shields [8] is contained in Theorem 2.4.3 in the monograph [2].

Theorem 2. *Suppose that* $T : [0, 1]^2 \longrightarrow [0, 1]$ *satisfies the following conditions:*

(i) $T(x, 0) = T(0, x) = 0$ *for all* x *in* $[0, 1]$*,*
(ii) $T(1, 1) = 1$*,*
(iii) T *is associative,*
(iv) T *is continuous.*

Then T *is a continuous t-norm on* $[0, 1]$.

Alsina, Frank and Schweizer raised the following question in [1] and repeated in [2]: Whether the continuity of T assumed in Theorem 2 can be weakened to continuity in each place?

3. The answer

Theorem 3. *Suppose that* $T : [0, 1]^2 \longrightarrow [0, 1]$ *satisfies the following conditions:*

(i) $T(x, 0) = T(0, x) = 0$ *for all* $x \in [0, 1]$*,*
(ii) $T(1, 1) = 1$*,*
(iii) T *is associative,*
(iv) T *is continuous in each place.*

Then T *is a continuous t-norm on* $[0, 1]$.

Proof. It is known that if $T : [0, 1]^2 \longrightarrow [0, 1]$ is both non-decreasing and continuous in each place then it is a continuous function [6,7]. So, thanks to Theorem 2, it suffices to show that T is non-decreasing in each place.

Firstly, we show that $T(x, 1) = T(1, x) = x$ for all $x \in [0, 1]$. Fix x in $[0, 1]$. Since $T(0, 1) = 0$, $T(1, 1) = 1$, and T is continuous in the first place, it follows that there exists some z such that $T(z, 1) = x$. Thus

$$T(x, 1) = T\big(T(z, 1), 1\big) = T\big(z, T(1, 1)\big) = T(z, 1) = x.$$

That $T(1, x) = x$ can be verified in a similar way.

Secondly, we show that $T(x, y) \leqslant y$. The conclusion is obvious if $x = 0$ or $x = 1$. So, we assume that $0 < x < 1$. Suppose on the contrary that there exists $a \in (0, 1)$ such that $T(x, a) > a$. Consider the continuous function $f : [0, 1] \longrightarrow \mathbb{R}$ given by $f(u) = T(x, u) - u$. Since $f(a) = T(x, a) - a > 0$ and $f(1) = x - 1 < 0$, there exists some $z \in (a, 1)$ such that $f(z) = 0$, hence $T(x, z) = z$. Since $T(z, 0) = 0$, $T(z, 1) = z$, and T is continuous in the second place, there exists some b such that $T(z, b) = a$. Then

$$T(x, a) = T\big(x, T(z, b)\big) = T\big(T(x, z), b\big) = T(z, b) = a,$$

a contradiction to that $T(x, a) > a$. This shows that $T(x, y) \leqslant y$.

A similar argument yields that $T(x, y) \leqslant x$, whence it follows that $T(x, y) \leqslant \min\{x, y\}$.

Thirdly, we show that T is non-decreasing in the second place. Suppose that $0 \leqslant y_1 < y_2 \leqslant 1$. Since $T(y_2, 0) = 0$, $T(y_2, 1) = y_2$, and T is continuous in the second place, there exists some z such that $T(y_2, z) = y_1$. Consequently, for each $x \in [0, 1]$,