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Discrete bipolar universal integrals

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Abstract

The concept of universal integral, recently proposed, generalizes the Choquet, Shilkret and Sugeno integrals. Those integrals admit a discrete bipolar formulation, useful in those situations where the underlying scale is bipolar. In this paper we propose the concept of discrete bipolar universal integral, in order to provide a common framework for bipolar discrete integrals, including as special cases the discrete Choquet, Shilkret and Sugeno bipolar integrals. Moreover we provide two different axiomatic characterizations of the proposed discrete bipolar universal integral.

Keywords: Choquet integral; Sugeno integral; Shilkret integral; Universal integral; Bipolar integral

1. Introduction

Recently, a concept of universal integral has been proposed [27]. The universal integral generalizes the Choquet integral [4], the Sugeno integral [36] and the Shilkret integral [34]. Moreover, in [24,25] a formulation of the universal integral with respect to a level dependent capacity has been proposed, in order to generalize the level-dependent Choquet integral [18], the level-dependent Shilkret integral [3] and the level-dependent Sugeno integral [30]. The Choquet, Shilkret and Sugeno integrals admit a discrete bipolar formulation, useful in those situations where the underlying scale is bipolar [12,13,17,19,21]. In this paper we introduce and characterize the discrete bipolar universal integral, which generalizes the discrete Choquet, Shilkret and Sugeno bipolar integrals.

Let us briefly describe the economic motivations of this paper. In the last three/four decades non-additive integrals—i.e. those integrals based on monotone measures, not necessarily additive—have been applied to many fields of Decision Analysis.

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For example, in the field of multiple-criteria decision aid (MCDA), the use of non-additive integrals (called fuzzy integrals) is nowadays pervasive [8,14]. The motivation is due, essentially, to the fact that non-additive integrals, when used as aggregation functions, allow for a natural representation of the interaction of criteria.

In decision making under risk and uncertainty for a large time, the dominant model has been the Expected Utility Theory (EUT) [39]. The EUT value function is based on the Lebesgue integral, but the additivity of this integral when applied to real choice (especially in economics) leads to unrealistic conclusions (see e.g. [1,5,23,37]). For these motivations the development of new theories, called non-EUT theories, and based on non-additive integrals has increased very fast (for a seminal survey we recommend [35]). In decision making under risk and uncertainty, the Choquet integral has firstly received an axiomatic characterization [32] and then has been successfully applied to economic models of decision: overall we remember the Choquet Expected Utility (CEU) of Schmeidler and Gilboa [7,33] and the Cumulative Prospect Theory of Tversky and Kahneman [38].

Very recently, one of the most interesting lines of research was concerned with the *bipolarity of choices*: the decision maker individuates a reference point and, then separates gains (alternatives greater than the reference point) from losses (alternatives smaller than the reference point); symmetric choices with respect to the reference point are considered. Regarding a general discussion on the use of bipolarity the reader is referred to [11,29], while regarding the generalization of well known integrals, used in MCDM, to the bipolar case, the reader is referred to [15,21]. Also in decision under risk and uncertainty, the necessity of new tools able to model the bipolarity has emerged [28,40]. In [22] the bipolar Choquet integral of Grabisch and Labreuche [13] has been used in order to obtain a bipolar generalization of CPT.

The paper is organized as follows. In Section 2 we introduce the basic concepts. In Section 3 we define and characterize the bipolar universal integral. In Section 4 we give an illustrative example of a bipolar universal integral which is neither the Choquet nor Sugeno or Shilkret type. Section 5 shows how the discrete universal integral can be also characterized in terms of a family of aggregation functions satisfying a set of desired axioms. Finally, in Section 6, we present conclusions.

2. Basic concepts

For the sake of simplicity, in this work we present the results in a multiple criteria decision making setting. Given a set of criteria $X = \{1, \dots, n\}$, an *alternative* $\boldsymbol{x}$ can be identified with a score vector $\boldsymbol{x} = (x_1, \dots, x_n) \in [-\infty, +\infty]^n$, being x_i the evaluation of $\boldsymbol{x}$ with respect to the ith criterion. Without loss of generality, in the following we consider the bipolar scale $[-1, 1]$ to expose our results, so that $\boldsymbol{x} \in [-1, 1]^n$. For all $\boldsymbol{x} = (x_1 \dots, x_n) \in [-1, 1]^n$, the set $\{i \in X \mid x_i \geqslant t\}$, $t \in [0, 1]$, is briefly denoted with $\{\boldsymbol{x} \geqslant t\}$. Similar meaning have the symbols $\{\boldsymbol{x} \leqslant t\}$, $\{\boldsymbol{x} > t\}$ and $\{\boldsymbol{x} < t\}$. For all $\boldsymbol{x}, \boldsymbol{y} \in [-1, 1]^n$ we say that $\boldsymbol{x}$ dominates $\boldsymbol{y}$ and we write $\boldsymbol{x} \succeq \boldsymbol{y}$, if $x_i \geqslant y_i$, $i = 1, \dots, n$. Let us consider the set $\mathcal{Q} = \{(A, B) \in 2^X \times 2^X \mid A \cap B = \emptyset\}$ of all disjoint pairs of subsets of X, see [12]. With respect to the binary relation $\precsim$ on $\mathcal{Q}$ defined as $(A, B) \precsim (C, D)$ iff $A \subseteq C$ and $B \supseteq D$, $\mathcal{Q}$ is a lattice, i.e., a partially ordered set in which any two elements have a unique supremum $(A, B) \vee (C, D) = (A \cup C, B \cap D)$ and a unique infimum $(A, B) \wedge (C, D) = (A \cap C, B \cup D)$. For all $(A, B) \in \mathcal{Q}$ the vector $\mathbf{1}_{(A,B)} \in [-1, 1]^n$ is the vector whose ith component equals 1 if $i \in A$, equals -1 if $i \in B$ and equals 0 else. A bipolar aggregation function $f : [-1, 1]^n \to [-1, 1]$ is a function such that $f(\boldsymbol{x}) \geqslant f(\boldsymbol{y})$ whenever $\boldsymbol{x} \succeq \boldsymbol{y}$ and $f(\mathbf{1}_{(X,\emptyset)}) = 1$, $f(\mathbf{1}_{(\emptyset,X)}) = -1$ and $f(\mathbf{1}_{(\emptyset,\emptyset)}) = 0$. We indicate with $\mathcal{A}_{[-1,1]^n}$ the set of aggregation functions on $[-1, 1]^n$.

Definition 1. A function $\mu_b : \mathcal{Q} \to [-1, 1]$ is a (normalized) bi-capacity [12,13,19] on X if

- $\mu_b(\emptyset, \emptyset) = 0$, $\mu_b(X, \emptyset) = 1$ and $\mu_b(\emptyset, X) = -1$;
- $\mu_b(A, B) \leqslant \mu_b(C, D)$ for all $(A, B), (C, D) \in \mathcal{Q}$ such that $(A, B) \precsim (C, D)$.

By the sake of simplicity, we shortly denote $\mu_b((A, B))$ with $\mu_b(A, B)$. Note that the specification of bi-capacities generally requires $3n - 1$ parameters. In order to reduce the number of these parameters (and complexity of bi-capacities), some authors have proposed the notion of k-additivity of bi-capacities by using the Möbius and bi-polar Möbius transform. For more details on this topic, the reader is referred to literature [12] and [6].