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Ortholinear and paralinear semi-copulas

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Abstract

A new method to construct semi-copulas is introduced. These semi-copulas are called *ortholinear* (resp. *paralinear*) semi-copulas and their construction is based on linear interpolation on segments that are perpendicular (resp. parallel) to the diagonal of the unit square. The classes of ortholinear and paralinear (quasi-)copulas are characterized as well.

Keywords: Semi-copula; Quasi-copula; Copula; Diagonal section; Opposite diagonal section; Linear interpolation

1. Introduction

Semi-copulas have recently gained importance in several areas of research, such as reliability theory, fuzzy set theory and multi-valued logic [2,12,19,21]. Special classes of semi-copulas, such as quasi-copulas and copulas, are widely studied. For instance, quasi-copulas appear in fuzzy set theoretical approaches to preference modelling and similarity measurement [6–8]. Due to Sklar's theorem [35], copulas have received ample attention from researchers in probability theory and statistics [23].

Recall that a semi-copula [16,18] is a function $S : [0, 1]^2 \to [0, 1]$ satisfying the following conditions:

(i) for any $x \in [0, 1]$, it holds that

$$S(x, 0) = S(0, x) = 0, \qquad S(x, 1) = S(1, x) = x;$$

(ii) for any $x, x', y, y' \in [0, 1]$ such that $x \leqslant x'$ and $y \leqslant y'$, it holds that $S(x, y) \leqslant S(x', y')$.

In other words, a semi-copula is nothing else but a binary aggregation function with neutral element 1.

The functions $T_{\mathbf{M}}$ and $T_{\mathbf{D}}$ given by $T_{\mathbf{M}}(x, y) = \min(x, y)$ and $T_{\mathbf{D}}(x, y) = \min(x, y)$ whenever $\max(x, y) = 1$, and $T_{\mathbf{D}}(x, y) = 0$ elsewhere, are examples of semi-copulas. Moreover, for any semi-copula S the inequality $T_{\mathbf{D}} \leqslant S \leqslant T_{\mathbf{M}}$

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holds. A semi-copula Q is a quasi-copula [20,22,28] if it is 1-Lipschitz continuous, i.e. for any $x, x', y, y' \in [0, 1]$, it holds that

$$\left|Q(x', y') - Q(x, y)\right| \leqslant \left|x' - x\right| + \left|y' - y\right|.$$

A semi-copula C is a copula [1,30] if it is 2-increasing, i.e. for any $x, x', y, y' \in [0, 1]$ such that $x \leqslant x'$ and $y \leqslant y'$, it holds that

$$V_C\left(\left[x, x'\right] \times \left[y, y'\right]\right) := C\left(x', y'\right) + C(x, y) - C\left(x', y\right) - C\left(x, y'\right) \geqslant 0.$$

$V_C([x, x'] \times [y, y'])$ is called the C-volume of the rectangle $[x, x'] \times [y, y']$. The copulas $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ with $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$, are respectively the greatest and the smallest copula, i.e. for any copula C, it holds that $T_{\mathbf{L}} \leqslant C \leqslant T_{\mathbf{M}}$.

The diagonal section of a $[0, 1]^2 \to [0, 1]$ function F is the function $\delta_F : [0, 1] \to [0, 1]$ defined by $\delta_F(x) = F(x, x)$. A diagonal function [13] is a function $\delta : [0, 1] \to [0, 1]$ satisfying the following conditions:

(D1) $\delta(0) = 0$, $\delta(1) = 1$;
(D2) δ is increasing;
(D3) for any $x \in [0, 1]$, it holds that $\delta(x) \leqslant x$;
(D4) δ is 2-Lipschitz continuous, i.e. for any $x, x' \in [0, 1]$, it holds that

$$\left|\delta\left(x'\right) - \delta(x)\right| \leqslant 2\left|x' - x\right|.$$

The functions $\delta_{T_{\mathbf{M}}}(x) = x$ and $\delta_{T_{\mathbf{L}}}(x) = \max(2x - 1, 0)$ are examples of diagonal functions. Moreover, for any diagonal function δ, it holds that

$$\delta_{T_{\mathbf{L}}} \leqslant \delta \leqslant \delta_{T_{\mathbf{M}}}.$$

The set of all diagonal functions will be denoted by $\mathcal{D}$. The set of all $[0, 1] \to [0, 1]$ functions that satisfy conditions (D1), (D2) and (D3) will be denoted by $\mathcal{D}_{\mathrm{S}}$; the subset of *absolutely continuous* functions in $\mathcal{D}_{\mathrm{S}}$ will be denoted by $\mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$. We will restrict our attention to the elements of $\mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$ when characterizing the class of ortholinear semi-copulas. Note that the k-Lipschitz continuity of a real function implies its absolute continuity [37], and hence, any diagonal function is absolutely continuous. The diagonal section of a copula C is a diagonal function. Conversely, for any diagonal function δ, there exists at least one copula C with diagonal section $\delta_C = \delta$. For instance, the copula C_δ defined by

$$C_\delta(x, y) = \min\left(x, y, \frac{\delta(x) + \delta(y)}{2}\right)$$

is the greatest symmetric copula with diagonal section δ [11,14,31].

Similarly, the opposite diagonal section of a $[0, 1]^2 \to [0, 1]$ function F is the function $\omega_F : [0, 1] \to [0, 1]$ defined by $\omega_F(x) = F(x, 1 - x)$. An opposite diagonal function [4] is a function $\omega : [0, 1] \to [0, 1]$ satisfying the following conditions:

(OD1) for any $x \in [0, 1]$, it holds that $\omega(x) \leqslant \min(x, 1 - x)$;
(OD2) ω is 1-Lipschitz continuous, i.e. for any $x, x' \in [0, 1]$, it holds that

$$\left|\omega\left(x'\right) - \omega(x)\right| \leqslant \left|x' - x\right|.$$

The functions $\omega_{T_{\mathbf{M}}}(x) = \min(x, 1 - x)$ and $\omega_{T_{\mathbf{L}}}(x) = 0$ are examples of opposite diagonal functions. Moreover, for any opposite diagonal function ω, it holds that

$$\omega_{T_{\mathbf{L}}} \leqslant \omega \leqslant \omega_{T_{\mathbf{M}}}.$$

The set of all opposite diagonal functions will be denoted by $\mathcal{O}$. The set of all $[0, 1] \to [0, 1]$ functions that satisfy condition (OD1) will be denoted by $\mathcal{O}_{\mathrm{S}}$; the subset of *absolutely continuous* functions in $\mathcal{O}_{\mathrm{S}}$ will be denoted by $\mathcal{O}_{\mathrm{S}}^{\mathrm{ac}}$. We will restrict our attention to the elements of $\mathcal{O}_{\mathrm{S}}^{\mathrm{ac}}$ when characterizing the class of paralinear semi-copulas. The