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New families of symmetric/asymmetric copulas

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Abstract

In 2004, Rodríguez-Lallena and Úbeda-Flores have introduced a class of bivariate copulas which generalizes some known families such as the Farlie–Gumbel–Morgenstern distributions. In 2006, Dolati and Úbeda-Flores presented multivariate generalizations of this class, also they investigated symmetry, dependence concepts and measuring the dependence among the components of each classes. In this paper, a new method of constructing binary copulas is introduced, extending the original study of Rodríguez-Lallena and Úbeda-Flores to new families of symmetric/asymmetric copulas. Several properties and parameters of newly introduced copulas are included. Among others, relationship of our construction method with several kinds of ordinal sums of copulas is clarified.

Keywords: Copulas; Dependence concepts; Measures of association; Tails

1. Introduction

Copulas are mathematical objects that fully capture the dependence structure among random variables. Since their introduction they have gained a lot of popularity in several fields like finance, insurance and reliability theory, etc. Copulas are a way of studying scale-free measures of dependence and also are a tool to build families of bivariate distributions with given margins, hence copulas are of interest to statisticians [8,16]. A copula is a function $C: [0,1]^2 \to [0,1]$ which satisfies:

(a) for every u, v in $[0,1]$, $C(u,0) = 0 = C(0,v)$ and $C(u,1) = u$ and $C(1,v) = v$;

(b) for every u_1, u_2, v_1, v_2 in $[0,1]$ such that $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$, $V_C(R) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geqslant 0$ (in other words, for all rectangles $R = [u_1,u_2] \times [v_1,v_2]$ whose vertices lie in $[0,1]^2$, $V_C(R) \geqslant 0$).

Copulas allow us to combine univariate distributions to obtain a joint distribution with a particular dependence structure, see the famous Sklar theorem [18]. As a result of Sklar's theorem, copulas link joint distribution functions to their one-dimensional margins.

In the literature we can see wide effort in construction of new copulas. Recall, for example, conic copulas [9], univariate conditioning method proposed in [11], UCS (univariate conditioning stable) copulas [4], several construction

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methods based on diagonal or horizontal (vertical) sections discussed in [5–7]. Rodríguez-Lallena and Úbeda-Flores [16] introduced a class of bivariate copulas of the form:

$$C_\lambda(u,v) = uv + \lambda f(u)g(v), \quad (u,v) \in [0,1]^2 \tag{1.1}$$

where f and g are two non-zero absolutely continuous functions such that $f(0) = f(1) = g(0) = g(1) = 0$ and the admissible range of the parameter λ is $[-1/\max(\alpha\gamma, \beta\delta), -1/\min(\alpha\delta, \beta\gamma)]$ where

$$\begin{aligned} &\alpha = \inf\{f'(u)\colon u \in A\} < 0, \qquad \beta = \sup\{f'(u)\colon u \in A\} > 0\\ &\gamma = \inf\{g'(v)\colon v \in B\} < 0, \qquad \delta = \sup\{g'(u)\colon u \in B\} > 0\\ &A = \{u \in [0,1]\colon f'(u) \text{ exists}\}, \qquad B = \{v \in [0,1]\colon g'(u) \text{ exists}\}. \end{aligned} \tag{1.2}$$

This class of copulas provides a method for constructing bivariate distributions with a variety of dependence structures and generalizes some known families such as the Farlie–Gumble–Morgenstern (FGM) distributions as well as the bivariate distributions introduced by Sarmanov in [17]. Dolati and Úbeda-Flores [3] provided procedures to construct parametric families of multivariate distributions which generalize (1.1). On the other hand, they have presented multivariate generalizations of this class. Also they studied the symmetry and dependence concepts, measuring the dependence among the components of each classes and provided several examples.

Kim et al. [10] generalized the method of Rodríguez-Lallena and Úbeda-Flores to any given copula. They presented a construction, considering an arbitrary given copula C, as below:

$$C^*_\lambda(u,v) = C(u,v) + \lambda f(u)g(v), \quad (u,v) \in [0,1]^2. \tag{1.3}$$

Here for any rectangle $R = [u_1,u_2] \times [v_1,v_2]$, λ should satisfy

$$\frac{-V_C(R)}{\triangle \times \max(\alpha\gamma, \beta\delta)} \leqslant \lambda \leqslant \frac{-V_C(R)}{\triangle \times \min(\alpha\gamma, \beta\delta)} \tag{1.4}$$

where $\alpha, \beta, \gamma, \delta$ are the same as in (1.2), $\triangle = (u_2 - u_1)(v_2 - v_1)$, u_1, u_2, v_1, v_2 are in $[0,1]$ such that $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$ and f, g are two non-zero absolutely continuous functions defined on $[0,1]$ such that $f(0) = f(1) = g(0) = g(1) = 0$. The method of Kim et al. gives only a poor sufficient condition for λ determination, it excludes positive λ's and also it is in general rather difficult to be applied.

Mesiar et al. [12] proposed a rather general construction method for bivariate copulas, generalizing some construction methods known from the literature. They have written Eq. (1.1) and Eq. (1.3) as below,

$$C^*_\lambda(u,v) = \max\bigl(0, C(u,v)\bigr) - \lambda \Pi\bigl(f(u), g(v)\bigr). \tag{1.5}$$

In this study we suggest how to generalize the study of Rodríguez-Lallena and Úbeda-Flores to new families of symmetric/asymmetric copulas. The extended new families have $n \geqslant 1$ parameters and so they are more flexible than the method of Rodríguez-Lallena and Úbeda-Flores. Thus they are able to model the more miscellaneous structures of dependency. Several selected dependence measures such as Kendall's tau, Spearman's rho, Gini's gamma and also the tails behaviors of these new families will be shown and at last several examples will be provided.

The rest of this paper is structured as follows. In Section 2 we introduce some new copulas families as a modification of the product copula. Section 3 discusses some properties of the new families. In Section 4 several examples will be investigated. Relationships to some other construction methods are given in Section 5. Section 6 summarizes the conclusion of our work.

2. Extensions of product copula

Let $f_i(u)$ and $g_i(v)$ for $i = 1, 2, \ldots, n$ be absolutely continuous real functions defined on $[0,1]$. Then we consider the function given by

$$C(u,v) = uv + \sum_{i=1}^{n} \lambda_i f_i(u) g_i(v), \quad n \geqslant 1, \tag{2.1}$$