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Negations on type-2 fuzzy sets

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Abstract

So far, the negation that usually has been considered within the type-2 fuzzy sets (T2FSs) framework, and hence T2FS truth values **M** (set of all functions from [0, 1] to [0, 1]), was obtained by means of Zadeh's extension principle and calculated from standard negation in [0, 1]. But there has been no comparative analysis of the properties that hold for the above operation and the axioms that any negation in **M** should satisfy. This suggests that negations should be studied more thoroughly in this context. Following on from this, we introduce in this paper the axioms that an operation in **M** must satisfy to qualify as a negation and then prove that the usual negation on T2FSs, in particular, is antimonotonic in **L** (set of normal and convex functions of **M**) but not in **M**. We propose a family of operations calculated from any suprajective negation in [0, 1] and prove that they are negations in **L**. Finally, we examine De Morgan's laws for some operations with respect to these negations.

Keywords: Type-2 fuzzy sets; Normal and convex functions; Negation; Strong negation; De Morgan's laws

1. Introduction

Type-2 fuzzy sets were introduced by L.A. Zadeh in 1975 [23], as an extension of type-1 fuzzy sets (FSs). Whereas an element's degree of membership in type-1 fuzzy sets is determined by a value in the interval [0, 1], an element's degree of membership in a type-2 fuzzy set (T2FS) is a fuzzy set in [0, 1], that is, a T2FS is determined by a membership function $\mu : X \to [0,1]^{[0,1]}$, where $\mathbf{M} = [0,1]^{[0,1]}$ is the set of functions from [0, 1] to [0, 1] (see [14,15,17]). In this paper, we will obtain both general results in T2FSs with degrees of membership in **M** and particular results for T2FSs with degrees of membership in the subset **L** of normal and convex functions of **M**.

In [19] Trillas studied and characterized the negations in [0, 1], showing that $n : [0,1] \to [0,1]$ is a strong negation if and only if there exists an order automorphism $g : [0,1] \to [0,1]$ such that $n(x) = g^{-1}(1 - g(x))$ for all $x \in [0,1]$. Bustince et al. introduced intuitionistic generators in order to build negations in Atanassov's intuitionistic fuzzy sets (IFSs, see [1]) in [3]; and Deschrijver et al. characterized strong intuitionistic negations based on strong negations in [0, 1] in [5]. Further characteristics for strong negations in IFSs and interval-valued fuzzy sets (IVFSs) were provided in [4]. In this paper, we define and study negations in the T2FS framework, where the main results and

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characterizations that we have obtained are for T2FSs with degrees of membership in **L**. Note that the formulas for modelling union, intersection and "negation" in T2FSs were obtained from Zadeh's extension principle. Any negation in [0, 1] could be used to obtain "negation" in T2FSs; so far, however, on the one hand, standard negation ($n = 1 - id$) has been considered to define the "negation" operation $\neg$ (see Definition 4) in T2FSs. We analyze in Section 3 whether this operation really does satisfy the negation axioms. On the other hand, in [6] and [7], other negations are considered, but without any analysis of their properties. In Section 3 we introduce the axioms that a function in **M** should satisfy to qualify as a type-2 negation and strong type-2 negation (see Definition 11). Additionally, we will build type-2 negations associated with negations in [0, 1] and study their properties.

The paper is organized as follows. In Section 2 we review some definitions, operations and properties on T2FSs reported in the recent literature (see [21]). In Section 3 we review (see [19,21]) the definitions and properties of negations in [0, 1] (which yield type-1 negations), and we introduce the axioms that a function in **M** should satisfy to qualify as a type-2 negation and strong type-2 negation. We then introduce operations N_n (see Definition 12), which are type-2 negations associated with negations in [0, 1], and we study their properties. Particularly, we analyze whether the operation $\neg$ satisfies the negation axioms in **M** or in **L**. Finally, in Section 4, we study De Morgan's laws with respect to the operation N_n of some binary operations in **M**.

2. Preliminaries

Throughout the paper we will denote by X a non-empty set that will represent the universe of discourse. Additionally, $\leqslant$ will stand for the order relation in the lattice of real numbers.

Definition 1. (See [22].) A type-1 fuzzy set (FS), A, is characterized by a *membership function* μ_A,

$$\mu_A : X \to [0,1],$$

where $\mu_A(x)$ is the degree of membership of an element $x \in X$ in the set A. The set of all fuzzy sets on X is denoted by $F(X)$.

Definition 2. (See [2,4,20].) An interval-valued fuzzy set (IVFS), A, is characterized by a *membership function* σ_A,

$$\sigma_A : X \to I = \{[a,b];\ 0 \leqslant a \leqslant b \leqslant 1\},$$

where $\sigma_A(x)$ is the degree of membership of an element $x \in X$ in the set A. The set of all interval-valued fuzzy sets on X is denoted by $IVFS(X)$.

The partial order $\leqslant_I$ can be defined in I such that $[a_1, a_2] \leqslant_I [b_1, b_2]$ if and only if $a_1 \leqslant b_1$ and $a_2 \leqslant b_2$. Using this order, we can naturally define a partial order on $IVFS(X)$: $\sigma_A \leqslant_I \sigma_B$ if and only if $\sigma_A(x) \leqslant_I \sigma_B(x)$ for all $x \in X$.

The notion of interval-valued fuzzy sets was introduced independently by Zadeh [23], Sambuc [16], Grattan-Guiness [10], Jahn [13], in 1975. And called interval-valued fuzzy sets since the early 80's (see [9,18]).

Definition 3. (See [15,17].) A type-2 fuzzy set (T2FS), A, is characterized by a *membership function*:

$$\mu_A : X \to \mathbf{M},$$

where $\mathbf{M} = Map([0,1],[0,1]) = [0,1]^{[0,1]}$. Therefore, $\mu_A(x)$ is a fuzzy set in the interval [0, 1], and is the *degree* of membership of an element $x \in X$ in the set A. $\mu_A(x)$ is represented by

$$\mu_A(x) = f_x$$

where

$$f_x : [0,1] \to [0,1].$$

The set of all type-2 fuzzy sets on X is denoted by $F_2(X)$. Fig. 1 shows a type-2 set, and Fig. 2 illustrates the degree of membership of an element x in the type-2 set.