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The Cauchy problem for complex fuzzy differential equations

Daria Karpenko [a], Robert A. Van Gorder [b,*], Abraham Kandel [c]

[a] Department of Mathematics, University of South Florida, United States
[b] Department of Mathematics, University of Central Florida, United States
[c] Department of Computer Science, University of South Florida, United States

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Abstract

We discuss the existence of a solution to the Cauchy problem for fuzzy differential equations that accommodates the notion of fuzzy sets defined by complex-valued membership functions. We first propose definitions of complex fuzzy sets and discuss entailed results which parallel those of regular fuzzy numbers. We then give two existence results relevant to the Cauchy problem for fuzzy differential equations in the case of bounded integral operators. These results require either Hölder continuous or Lipschitz continuous response functions.

Keywords: Complex-valued grades of membership; Complex fuzzy differential equations; Cauchy problem; Existence theorem; Uniqueness theorem

1. Introduction

The concept of complex fuzzy sets as sets with complex membership functions was first introduced by Ramot et al., who in [22] demonstrated the increased expressive power gained by endowing a set S with a complex membership function $\mu_S(x) = r_S(x) \cdot e^{i\omega_S(x)}$, where $r_S(x)$ and $\omega_S(x)$ are real-valued functions with r_S solely responsible for the fuzzy information and ω_S functioning as a phase term containing additional crisp information.

In [25], Tamir et al. pointed out the limitations of the mixed fuzzy and crisp definition of [22] and generalized it by allowing a fuzzy phase term. As illustrated with examples in [25], the advantage of this augmented definition of complex fuzzy sets is its ability to accommodate fuzzy cycles. Since then, of note is [26], where Tamir and Kandel propose an axiomatic framework for complex fuzzy logic and demonstrate its application to complex economic systems. We draw the readers attention to the difference between complex fuzzy sets and fuzzy complex numbers (a distinct concept); compare Buckley [5] (fuzzy complex numbers) and Tamir and Kandel [25] (complex fuzzy sets).

In this paper we provide a way of incorporating such complex fuzzy sets (henceforth referred to as complex fuzzy sets so as to avoid any confusion with regular fuzzy complex numbers) into the theory of fuzzy differential equations.

* Corresponding author. Tel.: +1 4078236284; fax: +1 4078236253.
E-mail address: rav@knights.ucf.edu (R.A. Van Gorder).

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This is accomplished by extending certain general results for metric spaces of fuzzy sets (see [8]) to spaces of complex fuzzy sets and using these to propose two Cauchy solution existence theorems for complex fuzzy real numbers.

The classical Peano theorem for nonfuzzy $\mathbb{R}^n$ states that the first order Cauchy problem

$$x'(t) = f\bigl(t, x(t)\bigr), \qquad x(t_0) = x_0 \tag{1.1}$$

has a solution if f is continuous. Much effort has been expended toward establishing Peano-like theorems for fuzzy differential equations and the Cauchy problem for fuzzy differential equations has been studied extensively: see [12] for results in differentiability and integrability properties and [13,18,20,24] for methods and results concerning Peano-like theorems for fuzzy differential equations. Kaleva in [13] concluded that, for fuzzy differential equations, just as for the classical case, continuity is a sufficient condition for the existence of a solution to the Cauchy problem; this conclusion, however, was disputed by Friedman et al. in [10]. Later, in Nieto [18], continuity and boundedness were claimed to be sufficient conditions for the existence of a solution to (1.1). However, Choudary and Donchev [7] showed that there was an error in the method of proof employed. In particular, they showed that the space of integral operators vital to the existence result of Nieto [18] is uniformly bounded, but not totally bounded. The existence proof of Nieto [18] relied on the total boundedness of this space. This is not to say that the claimed result is incorrect, only that the proof does not seem sufficient. Indeed, the result may very likely be correct, there is just no solid proof. Picard–Lindelöf type theorems for the Cauchy problem (1.1) were presented by Wu and Song [28].

For classical results about the spaces of fuzzy numbers in general, the reader is referred to [8]. In the other direction, for the generalization of the Cauchy problem for fuzzy differential equations to generalized metric spaces, Ref. [19] should be of interest.

In the following section we review the basic results and definitions from the theory of fuzzy differential equations and provide their corresponding extensions to complex fuzzy sets. The polar representation of the complex membership function is considered separately from the Cartesian representation due to its special periodic properties. We conclude with the presentation of two existence results for the Cauchy problem on the domain of complex fuzzy numbers. The first of these is similar in form to the result of Nieto [18]. This theorem is for f appropriately Hölder continuous in x. With this assumption, we bypass the need for total boundedness of the space of operators considered by Nieto; rather, uniform boundedness is sufficient. The second theorem is a Picard–Lindelöf type result, which requires Lipschitz continuity of f in x.

2. Background

Let $P_K(\mathbb{R}^n)$ denote the family of all nonempty convex compact spaces of $\mathbb{R}^n$. The Hausdorff metric for $A, B \in P_K(\mathbb{R}^n)$ is defined as

$$d(A, B) = \inf\bigl\{\varepsilon \bigm| A \subset N(B, \varepsilon) \text{ and } B \subset N(A, e)\bigr\}, \tag{2.1}$$

where $N(A, \varepsilon) = \{x \in \mathbb{R}^n \mid \|x - y\| < \varepsilon \text{ for some } y \in A\}$.

We consider complex fuzzy sets on $\mathbb{R}^n$, i.e. complex membership functions referred to as “pure complex fuzzy” in [25] (an expansion of the original definition in [22]), where Tamir et al. define a Cartesian and a polar representation of complex grades of membership. We consider the Cartesian definition first.

2.1. Cartesian representation of complex grades of membership

The complex membership function in [25], μ, is defined as

$$\mu(V, z) = \mu_R(V) + i\mu_I(z),$$

where V is to be interpreted as a set in a fuzzy set of sets and z as an element of V. This definition can be easily extended to $\mathbb{R}^n$: For $x \in \mathbb{R}^n$, let

$$f(x) = u(x) + iv(x),$$

where $u, v : \mathbb{R}^n \to [0, 1]$. For ease of notation, denote f by (u, v). Thus, f assigns to each $x \in \mathbb{R}^n$ a value in the unit square in $\mathbb{C}$, representing a complex grade of membership. Note that u, v considered individually define non-complex fuzzy sets in $\mathbb{R}^n$.