Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 245 (2014) 30–42
FUZZY sets and systems
www.elsevier.com/locate/fss

Existence and uniqueness results for fuzzy linear differential-algebraic equations

R. Alikhani [a,b], F. Bahrami [a], T. Gnana Bhaskar [b,*]

[a] *Department of Mathematics, University of Tabriz, Tabriz, Iran*
[b] *Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA*

Received 15 November 2012; received in revised form 26 January 2014; accepted 11 March 2014

Available online 17 March 2014

Abstract

We discuss the existence results for a fuzzy initial value problem of linear differential-algebraic equations and provide an explicit representation for the solution. A few illustrative examples are given.

Keywords: Differential-algebraic equations; Fuzzy differential equations; Initial value problems

1. Introduction

Differential equations occur naturally in the modeling of dynamical behavior of physical processes. If the states of the physical systems are in some ways constrained, for instance, by conservation laws such as Kirchhoff's laws in electrical networks, or by position constraints such as the movement of mass points on a surface, then the corresponding mathematical models contain algebraic equations to describe these constraints in addition to the differential equations that describe the dynamics of the system. Such systems, comprising of both differential and algebraic equations are called differential-algebraic equations (DAEs). Thus, the modeling of constrained mechanical systems, electrical circuits and chemical reaction kinetics, semi-discretization of systems of partial differential equations and singular perturbation problems see [6]), usually lead to systems of DAEs.

For a study of fundamental properties such as the canonical forms, existence and uniqueness theory etc., of a system of linear DAEs we refer to [11] and [22]. The study of DAE's is an active research area and there exist works that deal with different classes of DAEs such as partial DAEs [21], stochastic DAEs [24], functional DAEs [12], ill-posed DAEs [30]. For a numerical study of DAEs we refer to [6]. Recently, [26] dealt with the study of semi-explicit systems of nonlinear DAEs, where the coefficients of the system considered are estimated using stochastic collocation and Galerkin methods.

* Corresponding author. Tel.: +1 (321) 674 7213; fax: +1 (321) 674 7412.
E-mail addresses: alikhani@tabrizu.ac.ir (R. Alikhani), fbahram@tabrizu.ac.ir (F. Bahrami), gtenali@fit.edu (T. Gnana Bhaskar).

http://dx.doi.org/10.1016/j.fss.2014.03.006
0165-0114/

On the other hand, in order to obtain realistic models of some dynamical processes, it is often necessary to take into account imprecision, randomness or uncertainty in the measurement of underlying parameters (see [9,10,16,23]). Such models require the study of qualitative and numerical aspects of fuzzy differential equations, see [1,2,5,7,18,19].

To the best of our knowledge, a systematic study of existence theory of fuzzy DAEs is yet to be done. Initiating such a study in this work, we study the existence results for the following initial value problem associated with a system of fuzzy linear differential-algebraic equations:

$$E\dot{x}(t) = Ax(t) + f(t) \quad t \in \mathbb{I},$$
$$x(0) = \gamma, \tag{1}$$

where $x(t) = [x_1(t), \dots, x_n(t)]^T$, E and A are $n \times n$ real matrices, E is a singular matrix, $f \in C(\bar{\mathbb{I}}, \mathbb{R}^n)$ and $\gamma \in \mathbb{R}^n_{\mathcal{F}}$, $n \in \mathbb{N}$. Here the interval $\mathbb{I} = (0, T)$ for some $T > 0$, or $(0, \infty)$ and $\bar{\mathbb{I}}$ denotes the closure of $\mathbb{I}$.

In our work, we consider the general model in which the entries of the coefficient matrices are not necessarily positive. Our approach is based on the idea to separate the positive and negative entries by rearranging the unknowns in a suitable order. We replace the fuzzy system of order $n \times n$ by a crisp system of order $(2n) \times (2n)$ in which the positive and the negative entries are separated. We find the conditions that ensure that the solution obtained in the process is a valid fuzzy solution.

The paper is organized as follows. In Section 2, we introduce some basic notions of fuzzy numbers and describe the existence theory of the crisp solution for linear differential-algebraic equations. In Section 3, which is the main section of the paper, we study the existence of solution of the fuzzy initial value problem (1). A few illustrative examples are presented in Section 4.

2. Preliminaries

In this section we recall a few known results that are needed in our work.

The space of fuzzy numbers (see [13]), denoted by $\mathbb{R}_{\mathcal{F}}$, is the set of functions $u : \mathbb{R} \to [0, 1]$ that have the following properties:

(i) u is normal, i.e. there exists $t_0 \in \mathbb{R}$ such that $u(t_0) = 1$
(ii) u is fuzzy convex, i.e. $u(\lambda t_1 + (1-\lambda)t_2) \geqslant min\{u(t_1), u(t_2)\}$, for any $t_1, t_2 \in \mathbb{R}$, $\lambda \in [0, 1]$
(iii) u is upper semicontinuous
(iv) $[u]^0 = cl\{t \in \mathbb{R} \mid u(t) > 0\}$ is compact. (Here $cl\,A$ denotes the closure of A.)

For $0 < \alpha \leqslant 1$, α-level set of $u \in \mathbb{R}_{\mathcal{F}}$ is defined by

$$[u]^\alpha = \{t \in \mathbb{R} \mid u(t) \geqslant \alpha\}.$$

For any $\alpha \in [0, 1]$, $[u]^\alpha = [\underline{u}^\alpha, \bar{u}^\alpha]$ is a bounded closed interval. For $u, v \in \mathbb{R}_{\mathcal{F}}$, $\lambda \in \mathbb{R}$, we define the addition $u + v$ and scalar multiplication $\lambda \cdot u$ as:

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha \quad \text{and}$$
$$[\lambda \cdot u]^\alpha = \lambda[u]^\alpha,$$

where $[u]^\alpha + [v]^\alpha$ and $\lambda[u]^\alpha$ mean the usual addition of two subsets of $\mathbb{R}$ and the usual product between a scalar and a subset of $\mathbb{R}$ respectively (see [13,29]).

Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists a unique fuzzy number $w \in \mathbb{R}_{\mathcal{F}}$ such that $v + w = u$, then w is called the H-difference of u, v and is denoted by $u \ominus v$ (see [27]).

The Hausdorff distance between u and v is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$

$$D(u, v) = \sup_{\alpha \in [0,1]} max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\bar{u}^\alpha - \bar{v}^\alpha|\}.$$

The space $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.