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Law of large numbers for the possibilistic mean value

Pedro Terán

Escuela Politécnica de Ingeniería, Departamento de Estadística e I.O. y D.M., Universidad de Oviedo, E-33071 Gijón, Spain

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Abstract

A law of large numbers for the possibilistic mean value of a variable in a possibility space is presented. An example shows that the convergence in distribution (under a definition involving the possibilistic mean value) of the sample average to a variable with a certain distribution cannot be replaced, in general, by convergence either almost surely or in necessity. Even so, the usual presentation of the law of large numbers as a statement that holds 'in necessity' follows from this result.

Keywords: Convergence in distribution; Convergence in necessity; Law of large numbers; Possibilistic mean value; Possibility measure

1. Introduction

The law of large numbers (LLN) is one of the foundational results of probability theory, and it has been extended to possibilistic variables (also called fuzzy variables) in papers like [1,6,8,13]. The analogue of the expected value is then a modal value (i.e. a value with possibility 1), and traditionally it has been assumed that the modal value is unique, emphasizing the analogy with the probabilistic law of large numbers.

We set

$$S_n = n^{-1} \sum_{i=1}^{n} X_i$$

for a sequence of variables $\{X_n\}_n$ under consideration. In the probabilistic case, the law of large numbers is called strong or weak depending on whether it states the almost sure convergence

$$P\big(S_n \to E[X]\big) = 1,$$

or the convergence in probability

$$P\big(\big|S_n - E[X]\big| < \varepsilon\big) \to 1 \quad \text{for all } \varepsilon > 0.$$

This paper, however, starts at an alternative expression of the Weak LLN using convergence in distribution:

$$S_n \to E[X] \quad \text{in distribution.} \tag{1}$$

E-mail address: teranpedro@uniovi.es.

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Convergence in distribution is in general weaker than convergence in probability, but they are equivalent when the limit is non-random. Thus (1) is not weaker than the Weak LLN, as it might seem.

Further, convergence in distribution is commonly defined as the pointwise convergence of the distribution functions at all the continuity points of the limit, but other equivalent conditions are given by the portmanteau lemma (see e.g. [3, Theorem 2.1, p. 16]), in view of which (1) is tantamount to

$$E\left[f(S_n)\right] \to f(x) \quad \text{for all continuous bounded } f : \mathbb{R} \to \mathbb{R} \tag{2}$$

and $x = E[X]$. In this form, the law of large numbers reduces to a statement on the limit behaviour of the expected value of functions of S_n, saying that they all converge to the image of the expected value of X. That can be read as saying that the distribution of the sample averages S_n converges to the Dirac distribution $\delta_{E[X]}$.

Our second ingredient is the notion of a possibilistic mean value introduced by Carlsson and Fullér [4] (who acknowledge the prior work of Goetschel and Voxman [7]), a 'fuzzy' or 'possibilistic' version of the expected value for fuzzy information. The possibilistic mean value of a fuzzy number is very popular (Google Scholar finds over 450 citations to the Carlsson–Fullér paper) and easy to adapt to variables instead of fuzzy numbers. Indeed, let U be a fuzzy number, with α-cuts U_α for $\alpha \in (0, 1]$, and define

$$MV[U] = \int_0^1 \alpha(\inf U_\alpha + \sup U_\alpha)\, d\alpha.$$

Now let Π be a possibility measure on a measurable space $(\Omega, \mathcal{A})$, and $X : \Omega \to \mathbb{R}$ a measurable function which will be called a *variable*. Since

$$\Pi(A) = \sup_{x \in A} \pi(x)$$

for the possibility distribution $\pi(x) = \Pi(\{x\})$, the variable X induces a possibility distribution π_X given by $\pi_X(x) = \sup_{X(\omega)=x} \pi(\omega)$. Thus we can define

$$MV[X] := MV[\pi_X]$$

as soon as the integral in the right-hand side makes sense (even if π_X is not really a fuzzy number).

In general, there is not a law of large numbers in the sense that S_n does not converge in a way analogous to probabilistic 'almost sure' or 'in probability' convergences. To show that, we will construct an example of variables X and Y such that $\pi_X = \pi_Y$ but their sample averages converge to different limits or fail to converge.

The aim of this paper is to show that, even so, MV does fulfill a law of large numbers in the sense of 'convergence in distribution' outlined above. That is, for every bounded continuous function f we have

$$MV\left[f(S_n)\right] \to MV\left[f(Y)\right]$$

for some variable Y whose possibility distribution will be specified. By analogy to the probabilistic case, we can say that $S_n \to Y$ in distribution (under the possibilistic mean value). As will be shown, this mode of convergence implies the law of large numbers with convergence in necessity in e.g. [6].

Let us remark that, from the method of proof, it is very plausible that this form of the law of large numbers is valid as well for other functionals, not just MV.

2. Main result

Let us recall some basic notions, many of which are discussed in detail in e.g. [9]. A *triangular norm* $\top$ is an operation in [0, 1] which is associative, increasing as a bivariate function, and such that $x \top 1 = x$. It is called *Archimedean* if, for any $x \in (0, 1)$ and $\varepsilon > 0$, there exists an iterate $x \top \cdots \top x < \varepsilon$.

A *possibility measure* (e.g. [5]) is a set function Π on a measurable space $(\Omega, \mathcal{A})$ such that

$$\Pi(A) = \sup_{\omega \in A} \pi(\omega)$$

for some function $\pi : \Omega \to [0, 1]$, and such that $\Pi(\Omega) = 1$ (with the convention $\sup_{\omega \in \emptyset} \pi(\omega) = 0$). The set function