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Fuzzy Sets and Systems 187 (2012) 1–32

FUZZY sets and systems

www.elsevier.com/locate/fss

Preordered sets valued in a GL-monoid☆

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Received 21 October 2010; received in revised form 20 June 2011; accepted 20 June 2011
Available online 29 June 2011

Abstract

Let $M = (L, *)$ be a GL-monoid. An M-valued preordered set is an L-subset endowed with a reflexive and M-transitive L-relation, it is essentially a category enriched in a quantaloid generated by M. This paper presents a study of M-valued preordered sets with emphasis on symmetrization and the Cauchy completion. The main result states that symmetrization and the Cauchy completion of M-valued preordered sets commute up to a natural isomorphism.

Keywords: GL-monoid; Quantaloid; L-subset; L-relation; M-valued preordered set; M-valued set; Enriched category; Bimodule; Cauchy completion; Symmetrization

1. Introduction

The study of categorical foundations of fuzzy sets has been of intense interest in the fuzzy community since Zadeh [58] introduced the notion of fuzzy sets in 1965. The reader is referred to Gottwald [14] for a survey on this topic. One of the most attractive approaches is the theory of M-valued sets initiated in Höhle [21,22]. The key idea is to extend the theory of frame-valued sets to a theory of sets valued in a GL-monoid $M = (L, *)$.

The theory of frame-valued sets begins in the early seventies of the last century and has been developed by Higgs [20] and Fourman and Scott [12] (see also [37]). Let $\Omega = (H, \wedge)$ be a frame. An Ω-set is a pair (A, E), where A is a set and $E : A \times A \longrightarrow H$ is a map such that $E(a, b) = E(b, a)$ (symmetry) and $E(a, b) \wedge E(b, c) \leq E(a, c)$ (transitivity). It is easily seen that Ω-sets also satisfy the axiom: $E(a, b) \leq E(a, a) \wedge E(b, b)$ (strictness). The role of the strictness axiom becomes apparent when one wants to extend the notion of Ω-sets to non-symmetric and/or more general setting (see below). In particular, if a map $E : A \times A \longrightarrow H$ satisfies the axioms of strictness and transitivity (not necessarily symmetry), then the pair (A, E) is called a skew (non-symmetric) Ω-set [7]. The most important fact about Ω-sets is, perhaps, that the category of Ω-sets and morphisms is equivalent to the topos $\mathbf{Sh}(\Omega)$ of sheaves over Ω [12,13,20]. So, Ω-sets are a different formulation of sheaves over frames. Furthermore, Borceux and Cruciani demonstrated in [7] that the category of skew Ω-sets is equivalent to category of partially ordered sets in the topos $\mathbf{Sh}(\Omega)$.

Let L be a complete lattice. An L-subset [15] is a pair (X, λ), where X is a set and $\lambda : X \longrightarrow L$ is a map. Following [26,27,29], we interpret the value $\lambda(x)$ as the *extent of existence* of x.[1] There is a close connection between L-subsets

☆ This work is supported by Natural Science Foundation of China (11071174).
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[1] The reader is referred to [26,27,29] for more on the interpretation of $\lambda(x)$, and also to [12,13,48,54] for the logic and philosophy behind it. It should be mentioned that for fuzzy set theorists, $\lambda(x)$ is interpreted as the degree of membership of x [58].

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doi:10.1016/j.fss.2011.06.012

and frame-valued sets. To see this, assume that $\Omega = (H, \wedge)$ is a frame. Given an Ω-set (A, E), let $\mathbf{e}(E)(a) = E(a, a)$ for each $a \in A$. Then $(A, \mathbf{e}(E))$ is an H-subset. So, an Ω-set can be understood as an H-subset endowed with some extra structure (here, an H-valued equivalence relation). Here are the details. Given H-subsets (X, λ) and (Y, μ), an H-relation $R : (X, \lambda) \nrightarrow (Y, \mu)$ from (X, λ) to (Y, μ) is a map $R : X \times Y \longrightarrow H$ such that $R(x, y) \le \lambda(x) \wedge \mu(y)$ for all $x \in X$ and $y \in Y$. The composition $S \circ R : X \nrightarrow Z$ of H-relations $R : (X, \lambda) \nrightarrow (Y, \mu)$ and $S : (Y, \mu) \nrightarrow (Z, \nu)$ is given by

$$S \circ R(x, z) = \bigvee_{y \in Y} R(x, y) \wedge S(y, z).$$

For each H-subset (X, λ), the H-relation $\mathrm{id}_{(X,\lambda)} : (X, \lambda) \nrightarrow (X, \lambda)$ given by

$$\mathrm{id}_{(X,\lambda)}(x, y) = \begin{cases} \lambda(x), & x = y, \\ 0, & x \neq y \end{cases}$$

is clearly an identity w.r.t. the composition $\circ$. An H-relation $E : (X, \lambda) \nrightarrow (X, \lambda)$ is called an H-equivalence relation on (X, λ) if E satisfies the following axioms:

(i) $\mathrm{id}_{(X,\lambda)} \le E$, (**reflexivity**)
(ii) $E \circ E \le E$, (**transitivity**)
(iii) $E(x, y) = E(y, x)$ for all $x, y \in A$. (**symmetry**)

It is easily seen that a pair (A, E) is an Ω-set if and only if E is an H-equivalence relation on the H-subset $(A, \mathbf{e}(E))$. Similarly, (A, E) is a skew Ω-set if and only if E is a reflexive and transitive H-relation on $(A, \mathbf{e}(E))$. Thus, Ω-sets are equivalent to H-subsets endowed with H-valued equivalence relations; skew Ω-sets are equivalent to H-subsets endowed with reflexive and transitive H-valued relations.

On the other hand, given an H-subset (X, λ), $(X, \mathrm{id}_{(X,\lambda)})$ is trivially an Ω-set. This observation led Eytan [11] to identify H-subsets as a subcategory, denoted by $\mathbf{Fuz}(H)$, of the category Ω-**Set** of Ω-sets and morphisms. The relationship between $\mathbf{Fuz}(H)$ and Ω-**Set** is made clear in Pitts [43]: $\mathbf{Fuz}(H)$ is equivalent to the full subcategory of Ω-**Set** consisting of subconstant Ω-sets, and every Ω-set is a quotient of some subconstant Ω-sets. Therefore, the category of frame-valued sets provides, as argued in Höhle [26,27] and Vickers [54], an appropriate context to study L-subsets when L is a frame.

Since the introduction of quantales [40,45], people have tried to extend the theory of Ω-sets and sheaves over frames to a theory of quantale-valued sets and sheaves over quantales, see, e.g., [3,8,17,21,22,24,29,41,42]. A quantale is a complete lattice L together with an associative binary operation $*$ that distributes over arbitrary joins. Compared with the meet operation $\wedge$ in a frame, the semigroup operation $*$ in a quantale is neither commutative nor idempotent in general. So, in order to establish a theory of quantale-valued sets, one needs new techniques to cope with non-commutativity and non-idempotency. Early works in this direction, e.g., [3,8,41] (which is based on [42]), require that the quantale be right sided and idempotent. In that case, the notion of quantal sets is obtained by replacing the meet operation $\wedge$ in the definition of frame-valued sets by the semigroup operation $*$ with some necessary modification.

The requirement of idempotency is a serious drawback for fuzzy theorists because the semigroup operation in most of the quantales in fuzzy logic (e.g., complete BL-algebras [18]) is not idempotent, but commutative. To cope with the non-idempotency of the semigroup operation in the commutative setting, Höhle [21,22] proposed the notion of M-valued set in the case that $M = (L, *)$ is a GL-monoid (commutative and unital quantales satisfying the divisibility axiom, see Definition 2.2). We hasten to mention that frames and complete BL-algebras are all GL-monoids. An M-valued set is a pair (X, E), where X is a set and $E : X \times X \longrightarrow L$ is a map such that for all $x, y, z \in X$,

(i) $E(x, y) \le E(x, x) \wedge E(y, y)$, (**strictness**)
(ii) $E(x, y) * (E(y, y) \to E(y, z)) \le E(x, z)$, ($M$-**transitivity**)
(iii) $E(x, y) = E(y, x)$, (**symmetry**)

where $\to$ is a binary operation on L determined by the semigroup operation $*$.

M-valued sets have been employed to provide a categorical foundation for L-subsets in [14,29,49,50,57], and studied as presheaves over quantales in [17,21,22]. The notion has also been generalized to certain non-commutative settings in [17,24] and quite recently by a complete renunciation of the divisibility axiom to the general non-commutative and non-idempotent setting determined by involutive quantales [28].