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DAG representation of asymmetric independence models arising in coherent conditional possibility theory

Giuseppe Busanello [a,b], Davide Petturiti [b,c,*]

[a] *Istat – Istituto Nazionale di Statistica, via Balbo 16, 00184 Roma, Italy*
[b] *Dip. S.B.A.I., "La Sapienza" Università di Roma, via Scarpa 16, 00161 Roma, Italy*
[c] *Dip. Matematica e Informatica, Università degli Studi di Perugia, via Vanvitelli 1, 06123 Perugia, Italy*

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Abstract

In this paper we study the representation by means of an acyclic directed graph (DAG) of the independence model induced by a coherent T-conditional possibility (where T stands for the minimum or a strict t-norm). Such models are in general not closed under symmetric property, so we must rely on a proper asymmetric notion of vertex separation which produces structures closed under all graphoid properties and their reverses except for symmetry (namely, asymmetric graphoids). Focusing on this kind of models we present an efficient procedure to generate and represent them symbolically. We then introduce asymmetric Markov properties and prove their equivalence, providing in this way a method to extract the model encoded in a DAG. Finally, an algorithm to build a minimal I-map, given an ordering of the random variables, is drawn.

Keywords: Graphical models; Coherent conditional possibility; Independence models; Asymmetric graphoid; Acyclic directed graph; Fast closure; Asymmetric Markov properties; Possibilistic network

1. Introduction

Graphical models are fundamental tools in statistics and artificial intelligence which cope with the graphical representation of conditional independence statements induced by an uncertainty measure (see, e.g., [13,19,20,22,24,31,35,37,40–46,54]).

Historically, graphical models have been introduced in the probabilistic setting, but soon they have been extended also to other uncertainty frameworks such as Dempster–Shafer theory, credal set theory, lower and upper probability theory and possibility theory (see, e.g., [6,11,17,21,25,36,47,51]).

In any case, fixing an uncertainty calculus, to develop an ensuing graphical modeling, two notions are fundamental: *conditioning* and *independence*.

* Corresponding author.
E-mail addresses: giuseppe.busanello@sbai.uniroma1.it (G. Busanello), davide.petturiti@sbai.uniroma1.it (D. Petturiti).

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In possibility theory there are several proposals concerning conditioning (e.g., [23,26,29,30,33,56]) which basically emulate the Kolmogorovian approach to conditional probability. All these definitions of conditional possibility rely on the choice of a t-norm T and mostly present the conditional measure as a derived concept obtained from an unconditional one. In fact, the conditional possibility $\Pi(E|H)$ is essentially defined as a solution of the equation in x

$$\Pi(E \wedge H) = T\big(x, \Pi(H)\big),$$

so the computation of $\Pi(E|H)$ needs the knowledge of the joint and the marginal possibilities $\Pi(E \wedge H)$ and $\Pi(H)$. Moreover, it is immediate to see that, depending on the particular t-norm T, the above equation may have more than one solution (even requiring $\Pi(H) > 0$) or not a solution at all. Indeed, continuity of T only assures the solvability of previous equation so one needs to take care of non-uniqueness [26] or an extra constraint (e.g., the *minimum specificity principle* [29]) must be imposed in order to guarantee uniqueness.

For this reason, in this paper we adopt a different approach to conditioning in the spirit of de Finetti's conditional probability [28], dealing with the notion of *T-conditional possibility* directly defined as a function $\Pi(\cdot|\cdot)$ of two variables with a proper algebraic structure as domain and obeying to suitable axioms.

The main feature of this axiomatic definition is the concept of *coherence* (also inspired by the de Finetti's probabilistic analogous) which allows to handle *partial assessments* specified on *arbitrary* sets of conditional events. The coherence of an assessment is also necessary (and sufficient) to guarantee the extendability of the assessment to any finite superset of events, always preserving coherence [18]: this last property is particularly meaningful in inference processes [4,14].

In this environment a well-founded notion of conditional independence has been introduced in [17] for $T = \min$ and in [32] for strict t-norms, which is able to avoid pathological situations when logical constraints and extreme values 0 and 1 are involved. The main goal of this definition is to imply logical independence which is a prerogative of any notion of independence w.r.t. any uncertainty measure.

Anyway, the independence model induced by a coherent T-conditional possibility is in general not closed under symmetric property, hence common graphical tools used in the literature [35,37] cannot be used since they rely on symmetry. At this aim a proper asymmetric vertex separation criterion (namely, *L-separation criterion*) has been introduced in [46], in order to represent this type of models through an acyclic directed graph (DAG), coping also with the possible presence of logical constraints among the random variables. This separation condition determines independence structures closed with respect to all graphoid properties (and their reverse versions) except for symmetry, classified in [46] with the name of *asymmetric graphoids*, or *a-graphoids* for short.

In this paper we focus our attention on DAGs for representing an independence model compatible with a coherent T-conditional possibility in the cases $T = \min$ or a strict t-norm, closed under a-graphoid properties. In particular, here we assume no logical constraint is present between the random variables allowing a simplification of the L-separation condition that we will call *a-separation*.

The explicit generation and symbolic representation of an independence model closed with respect to a-graphoid properties is generally a hard problem since it requires an exponential amount of time and space. We face this task via the notion of *fast closure*, already studied in [3,41,42] in the case of semi-graphoids and in [1,3,42] for graphoids. In detail, two *generalized inference rules* for a-graphoids are presented allowing the introduction of an efficient algorithm to compute the fast closure.

Another ensuing problem is the extraction of the independence statements encoded in a DAG. This, in fact, can be done using the a-separation criterion and it requires to test all the triples of disjoint subsets of the vertex set. In order to test a smaller set of triples we introduce the *asymmetric Markov properties* (*global*, *local* and *pairwise*) and prove their equivalence for a-graphoids.

In particular, the equivalence between the global and the local asymmetric Markov properties together with the fast closure allow to design an efficient algorithm for the construction of a DAG representing the independence model, essentially tied to a given ordering of the random variables.

The paper is organized as follows. In Section 2 the fundamental notions of coherent T-conditional possibility and conditional independence are given. In Section 3 formal conditional independence models are presented while in Section 4 we introduce the asymmetric vertex separation criterion. In Section 5 we cope with the efficient generation and symbolic representation of the independence model by means of the fast closure and we give an algorithm based on two generalized inference rules. In Section 6 we define asymmetric Markov properties and prove their equivalence