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Strong integral of multifunctions relative to a fuzzy measure

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Abstract

In this paper we define and study an integral for measurable multifunctions taking values in the family of all nonvoid subsets of $[0, +\infty)$ with respect to a fuzzy measure. Unlike various ways of defining such an integral (e.g. Aumann way using different selections, Dunford way, Riemann way), we use the Sugeno method directly applied to multifunctions. The originality of our definition is that the value of the integral is a real scalar and not usually a set. Some classical integral properties, a convergence theorem and relationships with other integrals are presented.

Keywords: Fuzzy measure; Monotone measure; Fuzzy integral; Non-linear integral; Sugeno type integral; Measurable multifunction

1. Introduction

Since Aumann [4] introduced the integral of a multifunction, the theory of set-valued integrals has become of great interest due to its various applications in economics, control, probabilities. The theory of multifunctions (i.e. set-valued functions) has been studied by many authors both from the classical measure and the non-additive, set-valued or pseudo-analysis point of view (e.g. [3,11,15,20,22–24,28,29,38–40,45–47,50]). Measurable multifunctions are used, for instance, in economics [33,43] or in modeling birth-and-growth processes [2]. Generalizations of different classical inequalities [1,41] can be useful for studying the fluctuations in dynamic systems, economics and finance. Fuzzy integrals have many applications in different fields such as: decision-making processes, subjective evaluation processes, problems in fuzzy random variable, problems in diagnosing illnesses, problems in information fusion.

Integrals of multifunctions were defined in different ways:

– with selections [4,28,29,32,54–58];

– by the Rådström–Hörmander [48] embedding theorem [18];

– using simple multifunctions [34] or sequences of simple multifunctions [12,16,17,36] by a Dunford [21] type construction;

– using finite or countable sums that generalize the Riemann sums [5,6,8,25,31];

– via Choquet [10] or Sugeno [51] integrals [14,30,32,42,49,52].

In this paper, we define and study an integral for measurable multifunctions with respect to a fuzzy measure. This definition (of Sugeno type) is different from that of Aumann–Sugeno type in [54] and also different from that in [14],

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which is applied to the set-norm of multifunctions. Our definition may have the advantages that it is directly applied to the multifunctions, being computable and easily handled. The disadvantage is the fact that our definition is applied for multifunctions with values in $\mathcal{P}_0(\mathbb{R}_+)$, the space of all nonvoid subsets of $\mathbb{R}_+ = [0, +\infty)$. Our next goal will be to expand this definition to more general spaces. Some classical properties, a convergence theorem and relationships with other integrals are established. The novelty contributed here (and also in [14]) is that the integral of a multifunction is a real scalar and not usually a set. This type of integral can be useful in problems of synthetic quality evaluation, when the score function may be set-valued (i.e. for each quality factor there exist a multiple score or a set of estimations).

The paper is organized as follows: Section 1 is for introduction. In Section 2 some preliminaries are given. In Section 3 we define a fuzzy integral (via the Sugeno integral) of measurable multifunctions relative to a fuzzy measure. Some classical properties and a convergence theorem are obtained. Relationships with other integrals and some applications are shown in Section 4 and, ultimately, Section 5 is for conclusions.

2. Preliminaries

Let T be a nonvoid set, $\mathcal{C}$ a σ-algebra of subsets of T and $\mathcal{P}(\mathbb{R})$ the family of all subsets of the real space $\mathbb{R}$.

For a nonvoid subset X of $\mathbb{R}$ we shall denote by $\mathcal{P}_0(X)$ the family of all nonvoid subsets of X and by $\mathcal{P}_k(X)$ the family of all compact nonvoid subsets of X.

We denote $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$, where $\mathbb{N}$ is the set of all non-negative integers.

If (X, d) is a metric space, then for every $A, B \in \mathcal{P}_0(X)$, the Hausdorff distance between A and B is $h(A, B) = \max\{e(A, B), e(B, A)\}$, where $e(A, B) = \sup_{x\in A} d(x, B)$ is the excess of A over B and $d(x, B) = \inf_{y\in B} d(x, y)$ is the distance from x to B.

By a multifunction (or a set-valued function) from T to X we mean a function $F : T \to \mathcal{P}_0(X)$.

For a multifunction $F : T \to \mathcal{P}(\mathbb{R})$, we denote

$$F^{-1}(U) = \{t \in T \mid F(t) \cap U \neq \emptyset\}, \quad \forall U \in \mathcal{P}(\mathbb{R}).$$

Definition 2.1. A multifunction $F : T \to \mathcal{P}(\mathbb{R})$ is called *measurable* if

$$F^{-1}(U) \in \mathcal{C}, \quad \text{for every closed set } U \subseteq \mathbb{R}.$$

For more results on measurable multifunctions, the reader can see for instance, Castaing and Valadier [9].

Definition 2.2. A set function $\mu : \mathcal{C} \to [0, +\infty]$ is called

(i) *subadditive* if $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$;
(ii) *null-additive* if $\mu(A \cup B) = \mu(A)$, whenever $A, B \in \mathcal{C}$ and $\mu(B) = 0$;
(iii) *continuous from below* if $\mu(\bigcup_{n=0}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$, for every $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$, $A_n \subseteq A_{n+1}$, $\forall n \in \mathbb{N}$ (denoted $A_n \nearrow A$, where $A = \bigcup_{n=0}^{\infty} A_n$);
(iv) *continuous from above* if $\mu(\bigcap_{n=0}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$, for every $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$, $A_{n+1} \subseteq A_n$, $\forall n \in \mathbb{N}$ and there exists $n_0 \in \mathbb{N}$ so that $\mu(A_{n_0}) < +\infty$ (denoted $A_n \searrow A$, where $A = \bigcap_{n=0}^{\infty} A_n$);
(v) *continuous* if μ is continuous both from below and from above;
(vi) *diffused* if $\mu(\{t\}) = 0$, for every $t \in T$ so that $\{t\} \in \mathcal{C}$;
(vii) *a fuzzy measure* if μ is monotone and $\mu(\emptyset) = 0$.

Remark 2.3. According to Drewnowski [19], a subadditive fuzzy measure is called a submeasure.

Suppose $\mu : \mathcal{C} \to [0, +\infty]$ is a non-negative set function.

Definition 2.4. A property P regarding the points of T is said to hold μ *almost everywhere* (denoted by μ-a.e.) if there exists $A \in \mathcal{C}$ with $\mu(A) = 0$ such that P holds on $T\backslash A$.

Definition 2.5. A real function $f : T \to \mathbb{R}$ is called *measurable* if $f^{-1}(U) \in \mathcal{C}$, for every closed set $U \subseteq \mathbb{R}$.