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Submeasures on nuanced MV-algebras

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Abstract

This paper further investigates the properties of submeasures on n-nuanced MV-algebras introduced by the author in a previous work. We prove that there is a one-to-one correspondence between the set of submeasures on an n-nuanced MV-algebra and the set of submeasures on its MV-center. As a main result, we prove an extension theorem for submeasures on n-nuanced MV-algebras. This result generalizes the extension theorems which have been proved by Riečan for MV-algebras and by Georgescu for Łukasiewicz–Moisil algebras. The perfect NMVA$_n$ is defined and studied and the notion of a bounded local submeasure on a perfect NMVA$_n$ is introduced. It is proved that any bounded local submeasure on a perfect NMVA$_n$ L can be extended to a submeasure on L.

Keywords: Nuanced MV-algebra; Submeasure; Exhaustive property; Monotone set; Local submeasure; Extension theorem

1. Introduction

In 1940 Gr.C. Moisil introduced the 3-valued and 4-valued Łukasiewicz algebras [22] and in 1941 the n-valued Łukasiewicz algebras [23] with the intention of algebrizing Łukasiewicz's n-valued logic. An example of A. Rose from 1956 established that for $n \geqslant 5$ the Łukasiewicz implication can no longer be defined on a Łukasiewicz algebra. Algebraic models for the n-valued Łukasiewicz logic are the MV_n-algebras introduced by R. Grigolia in 1977. Nowadays, Łukasiewicz algebras are called Łukasiewicz–Moisil algebras and they have been deeply investigated in [2]. In fact, Moisil invented a distinct logical system, named Moisil's logic, and the n-valued Łukasiewicz–Moisil algebra is its algebraic counterpart. It was proved in [18] that the n-valued Łukasiewicz–Moisil algebras (LM$_n$-algebras for short) are polynomially equivalent to the MV$_n$-algebras for $n = 3$ and $n = 4$ and every MV$_n$-algebra can be endowed with a canonical structure of LM$_n$-algebra. The Łukasiewicz logic has the implication as its primary connector, while the Moisil logic is based on the idea of nuance, expressed algebraically by the Chrysippian endomorphisms. Due to Moisil's determination principle, an n-valued sentence is determined by its Boolean nuances, so one could say that Moisil's logic is derived from the classical logic by the idea of nuancing. This tight relationship is algebraically expressed by the fundamental adjunction between the categories of Boolean and Łukasiewicz–Moisil algebras. The notion of *n-nuanced* MV-algebras (NMVA$_n$ for short) was introduced in [16], extending both MV-algebras and n-valued Łukasiewicz–Moisil algebras. In the new structure the authors put together two approaches to

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multiple-valuedness: that of infinitely-valued Łukasiewicz logic and that of Moisil's n-nuanced logic. Unifying two important types of structures in the algebra of logic, the n-nuanced MV-algebra is the corresponding algebraic model of the n-nuanced logical system constructed from a given one using the idea of nuance. Algebraic models for logical systems proved to be very good tools for the study of logics using properties of their algebras or vice-versa, once their connection has been established. Apart from its relevance for mathematical logic, the study of n-nuanced MV-algebras is also motivated by their interesting structural and categorical properties, as well as by their model-theoretical aspects. The following considerations will clarify the position of the nuanced structures with respect to fuzzy set theory:

(i) The *nuancing*, represented by a family of $n-1$ unary operations called *Chrysippian endomorphisms*, is not the same process as the one of *grading truth* expressed by the *membership function* from the case of fuzzy sets [29]. The grading truth is completing in same way a truth structure with intermediate values, like taking the continuum $[0, 1]$, instead of $\{0, 1\}$, while by nuancing an element is reducible to its Chrysippian values. Although in the literature the degree of truth is sometimes called "nuance", the two notions are fundamentally different (see [24] for a detailed discussion on this distinction). Since the infinitely-valued Łukasiewicz logic is more related to the fuzzy approach, and Moisil's n-nuanced logic is more concerned with nuances of truth rather than truth degree, the n-nuanced MV-algebras can be viewed as an algebraic nuancing mechanism for MV-algebras [16].

(ii) The study of *linguistic hedges* is one of the most important application of fuzzy sets and they have been introduced by Zadeh in [30] and deeply investigated in [3–6,13,20,21,31–34]. According to [32], a *linguistic variable* is defined by a quintuple $(\mathcal{X}, T(\mathcal{X}), U, G, M)$, where $\mathcal{X}$ is the name of the variable, $T(\mathcal{X})$ is the *term-set* of $\mathcal{X}$, U is a universe of discourse, G is a *syntactic rule* which generates the terms in $T(\mathcal{X})$ and M is a *semantic rule* which associates to each linguistic value X its *meaning* $M(X)$ (a fuzzy subset of U). The meaning of a linguistic value X is characterized by a *compatibility function* $c : U \to [0, 1]$, where $c(u)$ is the compatibility of $u \in U$ with X. In fact, $c(u)$ characterizes various properties of the object u, such as "small, medium, big", etc. The meaning of a linguistic variable is modified by the linguistic hedges (or *linguistic modifiers*), such as "very, more, less, extremely", etc., which specify various nuances of properties. Since the compatibility function is in fact a membership function, as we mentioned in (i) the term of "nuance" is not related with the process of nuancing from Moisil's logic.

Probability theory on multiple-valued logic algebras has become a topic of increasing interest. States on MV-algebras have been defined in [25,26], while states on Łukasiewicz–Moisil algebras have been studied in [15]. In [10] a concept of state on n-nuanced MV-algebras has been developed and the continuous states were investigated based on the concept of convergence introduced in [9]. Dobrakov proved an extension theorem for submeasures on Boolean algebras and this result has been generalized for the case of MV-algebras in [27]. A similar result was proved for submeasures on Łukasiewicz–Moisil algebras in [17].

In this paper we further investigates the properties of submeasures on n-nuanced MV-algebras introduced in [10]. We show that there is a one-to-one correspondence between the set of submeasures on an n-nuanced MV-algebra and the set of submeasures on its MV-center. As a main result, we prove an extension theorem for submeasures on n-nuanced MV-algebras. This result generalizes the extension theorems which have been proved by Riečan for MV-algebras and by Georgescu for Łukasiewicz–Moisil algebras. These results combine the methods used in [17,27] and some proofs are reduced to the MV-center of an n-nuanced MV-algebra. The perfect NMVA_n is defined and studied and the notion of bounded local submeasure on a perfect NMVA_n is introduced. It is proved that any bounded local submeasure on a perfect NMVA_n L can be extended to a submeasure on L.

2. Preliminaries on n-nuanced MV-algebras

In what follows we give a few basic definitions and properties of n-nuanced MV-algebras. We will denote $J := \{1, \ldots, n-1\}$ with $n \in \mathbb{N}$, $n \geqslant 2$. For details regarding the properties of n-nuanced MV-algebras, convergences and states of these algebras we refer the reader to [9,10,16].

Definition 2.1. (See [16].) A *generalized De Morgan algebra* is a structure $(L, \oplus, \odot, N, 0, 1)$ of the type $(2, 2, 1, 0, 0)$ such that the following conditions are satisfied: