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Stratified categorical fixed-basis fuzzy topological spaces and their duality

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Abstract

For an abstract category **C**, a class $\mathcal{M}$ of **C**-monomorphisms and a fixed **C**-object L, we introduce stratified **C**-$\mathcal{M}$-L-spaces to be categorical counterparts of stratified fixed-basis fuzzy topological spaces in **C**, and consider their category **SC**-$\mathcal{M}$-L-**Top**. As two main results of this paper, it is shown that **SC**-$\mathcal{M}$-L-**Top** is dually adjoint to the comma category $L \downarrow$ **C**, and this adjunction can be restricted to a dual equivalence between the full category of $L \downarrow$ **C** with comma-spatial objects and the full category of **SC**-$\mathcal{M}$-L-**Top** with comma-sober objects. The present paper also describes applications and relationships of these results to stratified fixed-basis fuzzy topological spaces. In this respect, a considerable part of this paper is devoted to stratified L-quasi-topological spaces and their duality.

Keywords: Category theory; Topology; Categorical topology; Adjoint situation; Comma category; Lattice-valued topology; Fuzzy topology; Stratified fuzzy topology

1. Introduction

Since R. Lowen's modification [24] of fuzzy topology satisfying constants condition,[1] stratified lattice-valued topological spaces [16,18,29,33,45,46] and their generalizations [34,35,43,44] have been major issues in the field of fuzzy (lattice-valued or many-valued) topology. There are two main approaches to the stratified fixed-basis lattice-valued topological spaces in [16,18]. The first one assumes the constants condition (such spaces are called weakly stratified L-topological spaces in [16,18]), while the second one assumes the truncation condition (such spaces are called stratified L-topological spaces in [16,18]). L-quasi-topological spaces introduced in [32] form a general framework for the fixed-basis lattice-valued topological spaces including L-topological spaces in [16,18] as well. One may extend the notion of (weakly) stratified L-topological space to L-quasi-topological space in an obvious way.

Given an abstract category **C**, **C**-$\mathcal{M}$-L-spaces, what we mean by categorical fixed-basis fuzzy topological spaces in the title of this paper, have been put forward in [10] to be appropriate categorical counterparts of the fixed-basis fuzzy

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[1] The term "stratified" for this modification was later coined by P.-M. Pu and Y.-M. Liu [28].

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topological spaces in **C** that enable to carry over the famous Papert–Papert–Isbell adjunction **Top** $\dashv$ **Loc** [20,21,26] to a general adjunction **X** $\dashv$ $\mathbf{C}^{op}$ in which **X** is a suitable category of such counterparts. It is shown in [10] that the category **C**-$\mathcal{M}$-L-**Top** of **C**-$\mathcal{M}$-L-spaces provides an answer to the asked category **X** as well as a categorical framework for the fixed-basis fuzzy topology without needing order theory or algebra. In addition to the adjunction **C**-$\mathcal{M}$-L-**Top** $\dashv$ $\mathbf{C}^{op}$, the dual equivalence (also called duality [27]) between the full subcategory $(\mathcal{E}, \mathcal{M})$-$L$-**Spat**-**C** of **C** with L-spatial objects and the full subcategory **C**-$\mathcal{M}$-L-**SobTop** of **C**-$\mathcal{M}$-L-**Top** with L-sober objects is another central result in [10].

The present paper brings the formulation of stratified categorical fixed-basis fuzzy topological spaces and their duality into focus. Firstly, we define stratified **C**-$\mathcal{M}$-L-spaces (Definition 3.6) together with their category **SC**-$\mathcal{M}$-L-**Top**; secondly, we establish an adjunction **SC**-$\mathcal{M}$-L-**Top** $\dashv$ $(L \downarrow \mathbf{C})^{op}$ (Theorem 3.10) in which $L \downarrow \mathbf{C}$ is the category of objects under L, also the so-called comma category [25]; thirdly, we refine this adjunction to a duality between the full subcategory $(\mathcal{E}, \mathcal{M})$-**CMSpat**-$L \downarrow \mathbf{C}$ of $L \downarrow \mathbf{C}$ of all comma-spatial objects and the full subcategory **CMSob**-**SC**-$\mathcal{M}$-L-**Top** of **SC**-$\mathcal{M}$-L-**Top** of all comma-sober objects (Corollary 3.15). Many categories of (weakly) stratified lattice-valued topological spaces in [16,18,29,33,46] and stratified variety-based spaces in [34] are instances of **SC**-$\mathcal{M}$-L-**Top**. Thereby, the duality between $(\mathcal{E}, \mathcal{M})$-**CMSpat**-$L \downarrow \mathbf{C}$ and **CMSob**-**SC**-$\mathcal{M}$-L-**Top** is applicable to all of these categories. On the other hand, this duality cannot be employed in the case of the category **S**L-**QTop** of stratified L-quasi-topological spaces. To overcome this adverse situation, we invoke the duality between $(\mathcal{E}, \mathcal{M})$-$L$-**Spat**-**C** and **C**-$\mathcal{M}$-L-**SobTop**, and formulate a duality for **S**L-**QTop** with the category of L_*-semi-quantales (Corollary 4.38).

This paper has been prepared in four sections. After this introductory section, the next section overviews the category **C**-$\mathcal{M}$-L-**Top** and the aforementioned adjunction and duality for **C**-$\mathcal{M}$-L-**Top**, while Section 3 considers stratified **C**-$\mathcal{M}$-L-spaces, their category **SC**-$\mathcal{M}$-L-**Top**, the adjunction **SC**-$\mathcal{M}$-L-**Top** $\dashv$ $(L \downarrow \mathbf{C})^{op}$ and the duality between $(\mathcal{E}, \mathcal{M})$-**CMSpat**-$L \downarrow \mathbf{C}$ and **CMSob**-**SC**-$\mathcal{M}$-L-**Top**. Final section is devoted to stratified L-quasi-topological spaces and their duality with L_*-semi-quantales.

2. An overview of categorical fixed-basis fuzzy topological spaces

2.1. Definitions and examples

Let **C**, $\mathcal{M}$ and L denote an abstract category with set-indexed products, a class of **C**-monomorphisms and a fixed **C**-object, respectively. This will be assumed throughout this paper, unless further assumptions or restrictions are made on **C**, $\mathcal{M}$ and L. As the reference material for categorical notions and facts not given in this paper, we refer to [1,25].

A **C**-$\mathcal{M}$-L-space, which can be thought of as a fixed-basis fuzzy topological space in **C**, is a pair $(X, \tau \overset{m}{\to} L^X)$, consisting of a set X and an $\mathcal{M}$-morphism $\tau \overset{m}{\to} L^X$ (the so-called **C**-$\mathcal{M}$-L-topology on X). **C**-$\mathcal{M}$-L-spaces form a category **C**-$\mathcal{M}$-L-**Top** with morphisms all $(X, \tau \overset{m_1}{\to} L^X) \overset{f}{\to} (Y, \nu \overset{m_2}{\to} L^Y)$, where $f : X \to Y$ is a function such that there exists a (necessarily unique) **C**-morphism $r_f : \nu \to \tau$ making

$$\begin{array}{ccc} \nu & \xrightarrow{r_f} & \tau \\ {\scriptstyle m_2}\downarrow & & \downarrow{\scriptstyle m_1} \\ L^Y & \xrightarrow{f_L^{\leftarrow}} & L^X \end{array} \tag{2.1}$$

commute [8,10]. Here $f_L^{\leftarrow} : L^Y \to L^X$ is the unique **C**-morphism satisfying

$$\pi_x \circ f_L^{\leftarrow} = \pi_{f(x)} \tag{2.2}$$

for all $x \in X$, where $\pi_x : L^X \to L$ $(\pi_y : L^Y \to L)$ is the $x(y)$-th projection morphism for all $x \in X$ $(y \in Y)$.

Quasivarieties [10] supply a large inventory of examples of **C**-$\mathcal{M}$-L-**Top**. Varieties of Ω-algebras [34] and the categories **SQuant**, **SSQuant**, **USQuant**, **CGR**, **CQML**, **SFrm**, **Frm** of semi-quantales (shortly, s-quantales) [32], strong semi-quantales [7], unital semi-quantales (shortly, us-quantales) [32], complete groupoids [16], complete quasi-monoidal lattices [18], semiframes [29,33], frames [21] are known examples of quasivarieties.