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Fuzzy Sets and Systems 267 (2015) 18–30

FUZZY sets and systems

www.elsevier.com/locate/fss

# On diagonal completion of lattice-valued diagonal Cauchy spaces

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Received 12 February 2014; received in revised form 3 May 2014; accepted 5 May 2014

Available online 22 May 2014

**Abstract**

We define and study diagonal axioms for lattice-valued Cauchy spaces. A completion of a weakly diagonal lattice-valued Cauchy space is constructed, which is weakly diagonal and the coarsest among such completions. It is at the same time at least as fine as any weakly regular completion.


*Keywords:* $L$-Topology; $L$-Cauchy space; Diagonal condition; Completion

## 1. Introduction

Lattice-valued Cauchy spaces were first defined, for frames as lattices, in [14]. They generalize classical Cauchy spaces [22,19,26] to the lattice-valued case. If the lattice is given by a complete Boolean algebra, lattice-valued uniform convergence spaces [18] and hence also lattice-valued uniform spaces [6] induce underlying lattice-valued Cauchy spaces. For a suitable choice of the lattice, probabilistic Cauchy spaces [21,24] are examples of lattice-valued Cauchy spaces.

Later, in [3], a slight generalization of lattice-valued Cauchy spaces and regularity for these spaces was studied. Other generalizations are studied in [27] and [25].

In [14] a notion of completeness was defined and, if the lattice has a prime bottom element, a completion for a non-complete lattice-valued Cauchy space was constructed, provided the space satisfies a so-called completion axiom. This completion generalizes a corresponding construction for probabilistic Cauchy spaces [21,24]. A weaker form of completeness was defined in [25] but no completion was constructed in that paper. In [27], using the concept of stratified $(L, M)$-filter tower space, the completion of [14] was improved by avoiding the restriction on lattices with prime bottom element.

In this paper, we continue the study of completions of lattice-valued Cauchy spaces. We introduce two diagonal axioms for these spaces and generalize a completion that preserves such a diagonal property, given by Kent and

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http://dx.doi.org/10.1016/j.fss.2014.05.013

Richardson in [21], to the lattice-valued case. This shows that for a certain class of lattice-valued Cauchy spaces, completions can be constructed without any further assumptions on the underlying lattice.

## 2. Preliminaries

We consider in this paper frames, i.e. complete lattices, $L$, where finite meets distribute over arbitrary joins, i.e. where $\alpha \wedge \bigvee_{i\in I} \beta_i = \bigvee_{i\in I}(\alpha \wedge \beta_i)$ for all $\alpha, \beta_i \in L$ $(i \in I)$ and all index sets $I$. The bottom element of $L$ is denoted by $\bot$ and the top element by $\top$. The lattice operation are extended pointwise from $L$ to $L^X$, the set of all $L$-sets $a, b, c, \ldots$ on $X$. In particular, we define, for $A \subseteq X$, $\alpha_A \in L^X$ by $\alpha_A(x) = \alpha$ if $x \in A$ and $\alpha_A(x) = \bot$ for $x \notin A$.

A *stratified $L$-filter* on $X$ [7,9] is a mapping $\mathcal{F} : L^X \longrightarrow L$ with the properties (F1) $\mathcal{F}(\top_X) = \top$, $\mathcal{F}(\bot_X) = \bot$, (F2) $\mathcal{F}(a) \le \mathcal{F}(b)$ whenever $a \le b$, (F3) $\mathcal{F}(a) \wedge \mathcal{F}(b) \le \mathcal{F}(a \wedge b)$ for all $a, b \in L^X$ and (Fs) $\alpha \le \mathcal{F}(\alpha_X)$ for all $\alpha \in L$. The set of all stratified $L$-filters on $X$ is denoted by $\mathcal{F}_L^s(X)$. An example of a stratified $L$-filter is the *point $L$-filter* $[x]$ defined by $[x](a) = a(x)$ for all $a \in L^X$. The set $\mathcal{F}_L^s(X)$ is ordered pointwise, i.e. for $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$ we denote $\mathcal{F} \le \mathcal{G}$ if for all $L$-sets $a \in L^X$ we have $\mathcal{F}(a) \le \mathcal{G}(a)$. The set $\mathcal{F}_L^s(X)$ then has maximal elements which are called *stratified $L$-ultrafilters* and for each $\mathcal{F} \in \mathcal{F}_L^s(X)$ there is a finer stratified $L$-ultrafilter, see [7]. For a family of stratified $L$-filters $(\mathcal{F}_j)_{j\in J}$, the meet is then given by $\bigwedge_{j\in J} \mathcal{F}_j(a) = \bigwedge_{j\in J}(\mathcal{F}_j(a))$ for $a \in L^X$. In particular, we define for $\emptyset \ne A \subseteq X$, $[A] = \bigwedge_{x\in A}[x]$. Two stratified $L$-filters $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$ have an upper bound if and only if $\mathcal{F}(a) \wedge \mathcal{G}(b) = \bot$ whenever $a \wedge b = \bot_X$, see [9]. For a mapping $\varphi : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$ we define $\varphi(\mathcal{F}) \in \mathcal{F}_L^s(Y)$ by $\varphi(\mathcal{F})(b) = \mathcal{F}(\varphi^{\leftarrow}(b))$, where for $b \in L^Y$ it is defined $\varphi^{\leftarrow}(b)(x) = b(\varphi(x))$. Then $\varphi([x]) = [\varphi(x)]$ for every $x \in X$, $\varphi(\mathcal{F} \wedge \mathcal{G}) = \varphi(\mathcal{F}) \wedge \varphi(\mathcal{G})$ and, consequently, if $\mathcal{F} \le \mathcal{G}$ then also $\varphi(\mathcal{F}) \le \varphi(\mathcal{G})$. Moreover, for a further mapping $\psi : Y \longrightarrow Z$ we have $\psi \circ \varphi(\mathcal{F}) = \psi(\varphi(\mathcal{F}))$. For $\mathcal{G} \in \mathcal{F}_L^s(Y)$ we define $\varphi^{\leftarrow}(\mathcal{G}) : L^X \longrightarrow L$ by $\varphi^{\leftarrow}(\mathcal{G})(a) = \bigvee\{\mathcal{G}(b) : \varphi^{\leftarrow}(b) \le a\}$. Then $\varphi^{\leftarrow}(\mathcal{G}) \in \mathcal{F}_L^s(X)$ if and only if $\mathcal{G}(b) = \bot$ whenever $\varphi^{\leftarrow}(b) = \bot_X$ (see [10]). If $Y \subseteq X$ and $\iota : Y \hookrightarrow X$, $y \longmapsto y$ is the inclusion mapping, then we denote for $\mathcal{F} \in \mathcal{F}_L^s(X)$ its *trace on $Y$*, $\mathcal{F}_Y = \iota^{\leftarrow}(\mathcal{F})$, in case this is a stratified $L$-filter on $Y$. Also, for $\mathcal{G} \in \mathcal{F}_L^s(Y)$ we denote $[\mathcal{G}] = \iota(\mathcal{G}) \in \mathcal{F}_L^s(X)$.

For a set $J$ and $\mathcal{G} \in \mathcal{F}_L^s(J)$ and $\mathcal{F}^j \in \mathcal{F}_L^s(X)$ for all $j \in J$ we define the *stratified $L$-diagonal filter*, $\mathcal{G}(\mathcal{F}^{(\cdot)}) \in \mathcal{F}_L^s(X)$, by $\mathcal{G}(\mathcal{F}^{(\cdot)})(a) = \mathcal{G}(\mathcal{F}^{(\cdot)}(a))$ with $\mathcal{F}^{(\cdot)}(a)(j) = \mathcal{F}^j(a)$, see [12].

For notions of category theory we refer to [1].

## 3. Stratified $L$-Cauchy spaces and stratified $L$-limit spaces

A stratified $L$-Cauchy space [14] is a pair $(X, C)$ of a set $X$ and a mapping $C : \mathcal{F}_L^s(X) \longrightarrow L$ which satisfies the following axioms.

(LC1) $\forall x \in X$, $C([x]) = \top$.
(LC2) $\mathcal{F} \le \mathcal{G} \Longrightarrow C(\mathcal{F}) \le C(\mathcal{G})$.
(LC3) If $\mathcal{F} \vee \mathcal{G} \in \mathcal{F}_L^s(X)$, then $C(\mathcal{F}) \wedge C(\mathcal{G}) \le C(\mathcal{F} \wedge \mathcal{G})$.

A mapping $\varphi : (X, C) \longrightarrow (X', C')$ between two stratified $L$-Cauchy spaces $(X, C)$, $(X', C')$ is called *Cauchy-continuous* if for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ we have $C(\mathcal{F}) \le C'(\varphi(\mathcal{F}))$. The category which has as objects the stratified $L$-Cauchy spaces and as morphisms the Cauchy-continuous mappings is denoted by *SL-CHY*. The category *SL-CHY* is topological and cartesian closed [14]. In case $L = \{0, 1\}$ we can identify *SL-CHY* with the category *CHY* of Cauchy spaces [19,26].

A *stratified $L$-limit space* [11] is a set $X$ together with a limit map $\lim : \mathcal{F}_L^s(X) \longrightarrow L^X$ which satisfies the axioms

(LL1) $\lim[x](x) = \top$ for all $x \in X$,
(LL2) $\lim \mathcal{F} \le \lim \mathcal{G}$ whenever $\mathcal{F} \le \mathcal{G}$, $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$,
(LL3) $\lim \mathcal{F} \wedge \lim \mathcal{G} \le \lim(\mathcal{F} \wedge \mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$.

A mapping $\varphi : (X, \lim) \longrightarrow (X', \lim')$ between two stratified $L$-limit spaces is called *continuous* if for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ and all $x \in X$ we have $\lim \mathcal{F}(x) \le \lim' \varphi(\mathcal{F})(\varphi(x))$. The category with objects all stratified $L$-limit spaces and the continuous mappings as morphisms is denoted by *SL-LIM*. Also this category is topological and cartesian closed [11]. A stratified $L$-limit space is called a *T1-space* if $\lim[x](y) = \top$ implies $x = y$ and it is called a *T2-space* if $\lim \mathcal{F}(x) =$