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Short communication

On stratified L-convergence spaces: Fischer's diagonal axiom ☆

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Abstract

In the classical theory of convergence spaces, Fischer's diagonal axiom ensures that a generalized convergence space is topological. In this note, we present a lattice-valued Fischer diagonal axiom, and show that this axiom leads to the conclusion that a stratified L-convergence space is L-topological.

Keywords: Topology; Stratified L-convergence space; Stratified L-topological space; Fischer's diagonal axiom

1. Introduction

Stratified L-generalized convergence spaces, as a lattice-valued extension of generalized convergence spaces [20], were initiated by Jäger in [8]. Stratified L-convergence spaces or stratified L-ordered convergence spaces, as a subcategory of stratified L-generalized convergence spaces, were proposed by Li in [14] and Fang in [3]. In recent years both of the spaces received much attention, see [3,4,9–19,22].

Fischer's diagonal axiom [2], denoted as **(f)**, plays an essential role in the theory of classical convergence spaces. This axiom ensures that a generalized convergence space is topological. A lattice-valued Fischer diagonal axiom, denoted here in this text as **(Lfw)**, was first discussed by Jäger in [11]. Unfortunately, a stratified L-generalized convergence space (even a stratified L-convergence space) with **(Lfw)** may fail to be L-topological [11,16]. Later, another lattice-valued diagonal axiom, denoted as **(Lf)**, was proposed by Li and Jin in [16]. The axiom **(Lf)** ensures that a stratified L-generalized convergence space is L-topological. However, **(Lf)** uses a notion of neighborhood L-filters, which does not appear essentially in **(Lf)** in the case of $L = \{\bot, \top\}$. This shows that even the crisp **(Lf)** is different from **(f)**. The aim of the present note is to give a new lattice-valued diagonal axiom, such that: (1) it can be regarded as a common generalization of **(f)** and **(Lfw)**; (2) it does not resort to the notion of neighborhood L-filters; (3) it implies that a stratified L-convergence space is L-topological.

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The contents are arranged as follows. Section 2 fixes some notions and notations used in this note. Section 3 recalls some results about stratified L-topological convergence spaces. Section 4 presents the main results, i.e., an approximate Fischer diagonal axiom such that a stratified L-convergence space is L-topological.

2. Preliminaries

In this paper, if not otherwise specified, $(L, *, \top)$ is always a complete residuated lattice. That is, L is a complete lattice with a top element $\top$ and a bottom element $\bot$; $*$ is a binary operation on L such that (i) $(L, *, \top)$ is a commutative monoid; and (ii) $*$ distributes over arbitrary joins. Since the binary operation $*$ distributes over arbitrary joins, the function $\alpha * (-) : L \longrightarrow L$ has a right adjoint $\alpha \to (-) : L \longrightarrow L$ given by $\alpha \to \beta = \bigvee\{\gamma \in L : \alpha * \gamma \le \beta\}$. The binary operation $\to$ is called the residuation with respect to $*$. We collect here some basic properties of the binary operations $*$ and $\to$. They can be found in many places, for instance, [6,7,21].

Proposition 2.1. *For all $\alpha, \beta, \gamma, \delta \in L$ and all $\{\alpha_j\}_{j\in J}, \{\beta_j\}_{j\in J} \subseteq L$, we have*

(I1) $\bot * \alpha = \bot$ *and* $\top \to \alpha = \alpha$;
(I2) $\alpha \to \beta = \top \Leftrightarrow \alpha \le \beta$;
(I3) $\alpha * (\alpha \to \beta) \le \beta$ *and* $(\alpha \to \beta) * (\beta \to r) \le \alpha \to r$;
(I4) $\alpha \to (\beta \to r) = (\alpha * \beta) \to r = \beta \to (\alpha \to r)$;
(I5) $\alpha \le (\alpha \to \beta) \to \beta$;
(I6) $(\bigvee_{j\in J} \alpha_j) \to \beta = \bigwedge_{j\in J}(\alpha_j \to \beta)$;
(I7) $\alpha \to (\bigwedge_{j\in J} \beta_j) = \bigwedge_{j\in J}(\alpha \to \beta_j)$;
(I8) $\alpha \le \beta \Rightarrow \alpha \to \gamma \ge \beta \to \gamma$ *and* $\gamma \to \alpha \le \gamma \to \beta$;
(I9) $(\alpha \to \beta) * (\gamma \to \delta) \le (\alpha * \gamma) \to (\beta * \delta)$.

When L being a frame, we call an element γ a prime element in L if $\alpha \wedge \beta \le \gamma \Longrightarrow \alpha \le \gamma$ or $\beta \le \gamma$ for all $\alpha, \beta \in L$. Similarly, we call an element γ a prime element in L if $\alpha * \beta \le \gamma \Longrightarrow \alpha \le \gamma$ or $\beta \le \gamma$ for all $\alpha, \beta \in L$. It is easily seen that each element in a linearly ordered frame L is prime.

For a set X, the set L^X of functions from X to L with the pointwise order becomes a complete lattice. Each element of L^X is called an L-subset (or a fuzzy subset) of X. For any $\lambda \in L^X$, $\mathscr{K} \subseteq L^X$ and $\alpha \in L$, we denote by $\alpha * \lambda$, $\alpha \to \lambda$, $\bigvee \mathscr{K}$ and $\bigwedge \mathscr{K}$ the L-subsets defined by $(\alpha * \lambda)(x) = \alpha * \lambda(x)$, $(\alpha \to \lambda)(x) = \alpha \to \lambda(x)$, $(\bigvee \mathscr{K})(x) = \bigvee_{\mu\in\mathscr{K}} \mu(x)$ and $(\bigwedge \mathscr{K})(x) = \bigwedge_{\mu\in\mathscr{K}} \mu(x)$. Also, we make no difference between a constant function and its value since no confusion will arise.

Definition 2.2. (See [7].) An L-filter on a set X, is a function $\mathcal{F} : L^X \longrightarrow L$ with

(F0) $\mathcal{F}(\bot) = \bot$;
(F1) $\mathcal{F}(\top) = \top$;
(F2) $\forall \lambda, \mu \in L^X$, $\mathcal{F}(\lambda) \wedge \mathcal{F}(\mu) = \mathcal{F}(\lambda \wedge \mu)$.

An L-filter is said to be stratified if it also fulfills

(F3) $\mathcal{F}(\alpha * \lambda) \ge \alpha * \mathcal{F}(\lambda), \forall \alpha \in L, \lambda \in L^X$.

The set of stratified L-filters on X is denoted by $\mathcal{F}_L^s(X)$.

Example 2.3. (1) For any non-empty subset A of X, the function

$$[A] : L^X \longrightarrow L, [A](\lambda) = \bigwedge_{x\in A} \lambda(x)$$

is a stratified L-filter on X, called *the principal L-filter* generated by A.

(2) Let $\{\mathcal{F}_j | j \in J\} \subseteq \mathcal{F}_L^s(X)$. Then $\bigwedge_{j\in J} \mathcal{F}_j \in \mathcal{F}_L^s(X)$.