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Membership values in arrow categories

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Abstract

In this paper we consider an internal representation of the lattice of membership values used by the relations in an arbitrary arrow category. This is achieved by defining a more general construction that pairs regular with membership values in one object. We will show basic properties of this construction including its relationship to the lattice of membership values. This characterization of the lattice of membership values allows to specify and reason about the lattice elements as elements of an object of the category. Furthermore, this representation also leads to an abstract approach to higher-order fuzziness based on arrow categories.

Keywords: Arrow category; Goguen category; Dedekind category; Type-2 fuzzy relation

1. Introduction

Tarski's theory of relation algebras provides an algebraic framework for relations. The categorical counterpart of relation algebras is given by allegories or Dedekind categories [4,10,11]. The standard model of these categories is the category **Rel** of sets and relations, of course. However, there are many other models. For example, given a complete Heyting algebra one may form the category $\mathbf{Rel}(L)$ of sets and L-valued (or L-fuzzy) relations. Such a relation $R : A \to B$ is a function from $A \times B$ to L assigning to each pair in $A \times B$ a degree of membership in the relation R from Heyting algebra L. This category provides two extra notions compared to the category **Rel**. First of all, an isomorphic copy of the lattice of membership values L can be identified by considering so-called scalar relations (see Definition 8). In addition, an L-fuzzy relation can be crisp, i.e., the membership value of every pair is either 0 (smallest element of L) or 1 (greatest element of L). Obviously, the collection of crisp relations in $\mathbf{Rel}(L)$ is equivalent to **Rel**. Any crisp relation R from $\mathbf{Rel}(L)$ can be identified with a relation S from **Rel** by $(x, y) \in S$ iff $R(x, y) = 1$. Several abstract notions in Dedekind categories have been proposed in order to capture crispness [5,8,9]. Later it was shown that this property cannot be expressed in the language of allegories or Dedekind categories [16,20]. As a consequence the notions of arrow and Goguen categories were introduced adding two additional operations to the

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theory of Dedekind categories leading to a suitable definition of crispness in those categories. For this reason, arrow categories establish an abstract framework to reason about L-fuzzy relations. This theory and some weaker version were further investigated in [6,18,19,21].

In this paper we are interested in representing the membership values from L by certain relations in an arbitrary arrow category. As already mentioned above, the lattice L is isomorphic to the lattice of scalar relations and/or the lattice of ideal relations. In particular, if we consider scalars on a unit 1 (terminal object for maps), then all relations from 1 to 1 are scalars. However, this representation lacks two essential properties. First of all, scalars different from least and greatest scalar are neither total nor crisp. In this paper we are interested in a representation that characterizes an object so that its "elements" are isomorphic to L. We use the term "elements" here in the setting of categories and allegories, i.e., an element is a map from a unit to the object in question. In addition, we will require that those "elements" be crisp because of the inherent fuzziness in arrow categories. Secondly, the inclusion order of L-fuzzy relations restricted to scalar relations, i.e., an external relation on hom-sets, is isomorphic to the order of the lattice L. But there is no relation $E : 1 \to E$ representing the order of L, i.e., there is no internal representation. The latter is important for certain application such as fuzzy controllers where calculations on membership values have to be performed [20].

Besides the fact that it allows us to specify and reason about the lattice elements as elements of the category this also leads to a categorical approach to higher-order fuzziness based on arrow categories [22]. To explain this relationship we will consider L-fuzzy relations from A to B, i.e., a functions from $A \times B$ to L. A type-2 relation in this context is a relation that uses functions $L \to L$ as membership values, i.e., is a function $A \times B$ to $L \to L$ [2]. Since we know that $A \times B \to (L \to L)$ is actually isomorphic to $A \times (B \times L) \to L$, such a type-2 relation is isomorphic to a regular L-fuzzy relation from A to $B \times L$. Any abstract characterization of $B \times L$ will provide a notion of type-2 fuzziness based on the idea above. In order to characterize $B \times L$ one could try to characterize L and then use a relational product, or one could directly characterize $B \times L$ for any given B and use the unit 1 in order to obtain L. We will use the first approach in this paper.

The remainder of this paper is organized as follows. In Section 2 we will provide the required background on Dedekind and arrow categories. Section 3 is the main section of this paper. After providing the abstract definition of the extension $A^\sharp$ of an object A we will show that the arrow category **Rel**(L) of L-fuzzy relations always has extensions. Then we will provide some basic properties of this construction. In addition, we will give four examples verifying that the axioms of an extension are independent. The first of the two main theorems of this paper shows that the extension $A^\sharp$ is isomorphic to the relational product $A \times 1^\sharp$. The second main theorem establishes the fact that the lattice of elements of $1^\sharp$ is isomorphic to the lattice of scalar relation, i.e., that it is isomorphic to L in the case of L-fuzzy relations. Finally, Section 4 provides a conclusion and an outlook to future work.

2. Dedekind and arrow categories

We want to introduce some basic notions from lattice, category and allegory theory that will be used in this paper. For all properties not introduced here we refer the reader to [1] and [4].

We will write $R : A \to B$ to indicate that a morphism R of a category $\mathcal{R}$ has source A and target B. We will use ; to denote composition in a category with order of composition from left to right, i.e., $R; S$ means R first, and then S. The collection of all morphisms $R : A \to B$ is denoted by $\mathcal{R}[A, B]$. The identity morphism on A is written as $\mathbb{I}_A$.

A lattice is a structure $(L, \sqcup, \sqcap)$ with meet $\sqcap$ and join $\sqcup$. The order is denoted by $\sqsubseteq$. A lattice is called complete iff for every subset $M \subseteq L$ (including $\emptyset$) the least upper bound $\bigsqcup M$ and the greatest lower bound $\bigsqcap M$ of M in L exist. Notice, that a complete lattice has a least element $0 = \bigsqcup \emptyset = \bigsqcap \mathcal{L}$ and a greatest element $1 = \bigsqcap \emptyset = \bigsqcup \mathcal{L}$. A distributive lattice L is called a complete Heyting algebra iff L is complete and $x \sqcap \bigsqcup M = \bigsqcup_{y \in M}(x \sqcap y)$ holds for all $x \in L$ and $M \subseteq L$. Notice also that complete Heyting algebras have relative pseudo complements, i.e., for each pair $x, y \in L$ there is a greatest element $x \to y$ with $x \sqcap (x \to y) \sqsubseteq y$. Throughout this paper we will use the abbreviation $x \leftrightarrow y$ for $x \to y \sqcap y \to x$.

Now we want to recall some fundamentals on Dedekind categories [10,11]. Categories of this type are called locally complete division allegories in [4].

Definition 1. A Dedekind category $\mathcal{R}$ is a category satisfying the following: