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Fuzzy pseudometric spaces vs fuzzifying structures ☆

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Abstract

In this paper, we apply the representation theorem established in Mardones-Pérez and de Prada Vicente (2012) [12] to define and study the degree in which some topological-type properties of fuzzy pseudometric spaces are fulfilled. Fuzzifying structures which appear naturally are also investigated, and the relation between these structures and fuzzy pseudometric spaces is explored.

Keywords: Pseudometric; Uniformity; Fuzzifying uniformity; Fuzzifying topology; Fuzzy (pseudo)metric space; Convergence; Cauchy filters; Compactness; Precompactness; Completeness; Level theory

0. Introduction

Inspired by the notion of probabilistic metric spaces [17], Kramosil and Michalek [11] introduced in the seventies a notion of fuzzy (pseudo)metric in which the triangle inequality was controlled by a measurable real-valued function. A drawback in that definition (as it was the case in the original definition of probabilistic metric spaces, i.e. Menger spaces [13]) is that the associated convergence might be a non-topological one. This disadvantage can be taken away if the measurable function is substituted by a continuous or left-continuous t-norm, or more generally by a T-norm satisfying the condition $1 = \bigvee_{x\in(0,1)} T(x,x)$ (see [14,18]).

This last variant of the original definition of Kramosil and Michalek's fuzzy (pseudo)metric spaces was considered in a recent paper [12] under the name KM-fuzzy (pseudo)metric spaces. In that paper, any KM-fuzzy (pseudo)metric space, endowed with the strongest t-norm, was identified with a certain $[0, 1)$-indexed family of classical (pseudo)metrics.

Our main goal with this paper is twofold: first, to introduce and study the degree in which some topological and uniform properties of KM-fuzzy pseudometric spaces are satisfied and second, to establish some relations between KM-fuzzy pseudometric spaces and some particular fuzzy structures which appear naturally, the so-called fuzzifying structures. Both objectives are achieved as an application of the representation theorem given in [12].

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1. Preliminaries

1.1. KM-fuzzy pseudometric spaces

A *pseudometric* on a set X is a non-negative real-valued map d on $X \times X$ satisfying for all $x, y, z \in X$: (d1) $d(x, x) = 0$; (d2) $d(x, y) = d(y, x)$; (d3) $d(x, z) \leq d(x, y) + d(y, z)$. If d satisfies the additional condition (d4) $d(x, y) = 0 \Longrightarrow x = y$, then d is a metric.

A *t-norm* T is a binary operation on $[0, 1]$ satisfying for all $a, b, c, d \in [0, 1]$ the following properties: (T1) $T(c, d) \leq T(a, b)$ for $c \leq a, d \leq b$; (T2) $T(a, b) = T(b, a)$; (T3) $T(a, 1) = a$; (T4) $T(T(a, b), c) = T(a, T(b, c))$. Observe that $T(a, 0) = 0$ and $T(a, b) \leq a \wedge b$ for any $a, b \in [0, 1]$. A t-norm T_1 is *weaker* than another t-norm T_2 (T_2 is *stronger* than T_1) if $T_1(a, b) \leq T_2(a, b)$, for any $a, b \in [0, 1]$. Note that $T = \wedge$ is the strongest t-norm.

By a *KM-fuzzy pseudometric space* we mean a triple (X, m, T) where X is a nonempty set, T is a left-continuous t-norm and $m : X \times X \times [0, \infty) \to [0, 1]$ is a map satisfying for all $x, y, z \in X$ and $t, s \in [0, \infty)$ the following conditions: (FM1) $m(x, y, 0) = 0$; (FM2) $m(x, x, t) = 1$ for any $t > 0$; (FM3) $m(x, y, t) = m(y, x, t)$; (FM4) $T(m(x, y, t), m(y, z, s)) \leq m(x, z, t + s)$; (FM5) $m(x, y, -) : [0, \infty) \to [0, 1]$ is left-continuous; (FM6) $\lim_{t\to\infty} m(x, y, t) = 1$. The map m is called a *KM-fuzzy pseudometric*. From (FM2) and (FM4) it follows that $m(x, y, -) : [0, \infty) \to [0, 1]$ is non-decreasing. Moreover, (X, m, T) is said to be a *KM-fuzzy metric space* if, besides (FM1)–(FM6), the map m also satisfies (FM7) $m(x, y, t) = 1, \forall t > 0 \Longrightarrow x = y$.

Example 1.1. Associated with any (pseudo)metric space (X, d) (see [4], cf. [17, Theorem 5.1]), there is a KM-fuzzy (pseudo)metric space (X, m_d, T) (called the standard KM-fuzzy (pseudo)metric space), defining m_d for any $x, y \in X$ and $t \in [0, \infty)$ as

$$m_d(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{if } t > 0 \\ 0, & \text{if } t = 0. \end{cases}$$

In [12] KM-fuzzy pseudometric spaces under the t-norm $\wedge$ were identified with certain families of pseudometrics. We record the main notions and results involved in that correspondence.

Let $I \subseteq \mathbb{R}$ and Z be a set. A family of real-valued maps $\{d_i : Z \to \mathbb{R} : i \in I\}$ will be called *lower semicontinuous* if for any $i \in I$, $d_i = \bigwedge_{j>i} d_j$.

A KM-fuzzy pseudometric space (X, m, T) will be called *pseudometrically generated* if there exists a lower semicontinuous family of pseudometrics $\{d_a : a \in [0, 1)\}$ on X, such that for any $x, y \in X$ and $t \in [0, \infty)$,

$$m(x, y, t) = \bigvee \{a \in [0, 1) : d_a(x, y) < t\}.$$

Let $m : X \times X \times [0, \infty) \to [0, 1]$ be a KM-fuzzy pseudometric and for any $a \in [0, 1)$ let the map $m_a : X \times X \to [0, \infty)$ be defined as:

$$m_a(x, y) = \bigvee \{t \in [0, \infty) : m(x, y, t) \leq a\}, \quad \text{for any } x, y \in X. \tag{$\star$}$$

The family $\{m_a : a \in [0, 1)\}$ is lower semicontinuous, and hence, non-decreasing.

Fact 1.2. *(See [12, Lemma 3.5].) Let (X, m, T) be a KM-fuzzy pseudometric space. For any $a \in [0, 1)$, $x, y \in X$ and $t \in [0, \infty)$, the following equivalence holds:*

$$m(x, y, t) \leq a \iff t \leq m_a(x, y).$$

The maps $\{m_a : a \in [0, 1)\}$ become pseudometrics if and only if m is a KM-fuzzy pseudometric under the t-norm $\wedge$. Therefore, we have the following Representation theorem:

Fact 1.3. *(See [12, Theorems 3.12 and 3.13].) Any KM-fuzzy pseudometric space $(X, m, \wedge)$ is pseudometrically generated by the family of pseudometrics $\{m_a : a \in [0, 1)\}$, i.e.,*

$$m(x, y, t) = \bigvee \{a \in [0, 1) : m_a(x, y) < t\}.$$