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Fuzzy Sets and Systems 265 (2015) 1–23

FUZZY sets and systems

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# On fuzzy solutions for heat equation based on generalized Hukuhara differentiability

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Received 22 January 2014; received in revised form 7 November 2014; accepted 10 November 2014

Available online 15 November 2014

**Abstract**

In this paper a fuzzy heat equation with fuzzy initial values is considered. The concept of generalized Hukuhara differentiation is interpreted thoroughly in the univariate and multivariate cases, and also several properties for generalized Hukuhara differentiability are obtained on the topics, such as switching point, the univariate and multivariate fuzzy chain rules, fuzzy mean value theorem, among others. The objective of this paper is to prove the uniqueness of a solution for a fuzzy heat equation and show that a fuzzy heat equation can be modeled as two systems of fuzzy differential equations by considering the type of differentiability of solutions. Finally, some examples show the behavior of the solutions obtained.


*Keywords:* Generalized Hukuhara differentiability; Fuzzy partial differential equation; Fuzzy heat equation; Multivariate fuzzy chain rule; Mean value theorem; The maximum principle

## 1. Introduction

Partial differential equations are usually used to construct a multidimensional mathematical models not only of basic physical phenomena, but also various problems in social science. Many important dynamical systems in the real world can be described by partial differential equations, especially heat equations. The fuzzy set theory is a powerful tool for modeling uncertain problems. In heat equations, these vagueness may be appearing in each part of the heat equation like initial condition, boundary condition, etc. So solving heat equations in the sense of real conditions leads to the use of interval or fuzzy calculations.

The first definition of partial differential equation for the fuzzy-valued function was given by Buckley and Feuring [7] and it has been studied in several works. Allahviranloo in [1] proposed difference method for solving fuzzy partial differential equations (FPDEs). Also the Adomian decomposition method was studied for finding the approximate

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http://dx.doi.org/10.1016/j.fss.2014.11.009
0165-0114/

solution of the fuzzy heat equation in [2]. In [15], the authors considered application of fuzzy partial differential equations obtained through fuzzy rule-based systems. Furthermore, Oberguggenberger described weak and fuzzy solutions for FPDEs [19] and Chen et al. presented a new inference method with applications to FPDEs [8]. See [11] for interpretation of used FPDEs to modeling hydrogeological systems. Also studying heat, wave and Poisson equations with uncertain parameters can be found in [6].

In physical terms diffusion problem involves motion or transport of particles (ions, molecules, etc.) from areas of higher concentration to areas of lower concentration. Simple examples are the spread of a drop of ink into the water and the melting of an ice cube. The study of heat conduction in a rod produces the prototypical diffusion equation [13]. A fuzzy heat equation models the flow of heat in a rod that is insulated everywhere except at its two ends.

The purpose of the present paper is two topics. One of our intentions is to prove the uniqueness of the solution of a fuzzy heat equation. To achieve this purpose, we first prove some properties for generalized Hukuhara differentiability and generalized Hukuhara partial differentiability and using these properties we prove the uniqueness of the solution for a fuzzy heat equation. The other aim is solving a fuzzy heat equation under generalized Hukuhara differentiability. For this purpose, we convert a fuzzy heat equation into second order fuzzy boundary differential equation, then we use integrating factor method for solving this equation.

The paper organized as follows. Section 2 introduces the basic concept of generalized Hukuhara derivative and we also prove some properties of the generalized Hukuhara derivative. We study a fuzzy partial differential equation using generalized Hukuhara differentiability and we prove some properties for this concept of differentiability in Section 3. In Section 4, we define a fuzzy heat equation under generalized Hukuhara partial differentiability and using a maximum principle we prove the uniqueness of a solution for a fuzzy heat equation. Moreover according to the type of differentiability, solutions of fuzzy heat equations are investigated in different scenarios. In Section 5, some examples are given to show the efficiency of the proposed method and conclusions are drawn in Section 6. Finally in Appendix A, integrating factor method is presented to solve fuzzy boundary value differential equation.

## 2. Generalized Hukuhara derivative

Historically, the Hukuhara difference and derivative of a set-valued function are introduced in [14], that was the starting point of the topic in set-valued differential equations (see e.g. [16,20,4]). To overcome some shortcomings of this approach, Bede and Gal introduced the weakly generalized differential of a fuzzy number valued function. Also, they presented the sense of strongly generalized differentiability for fuzzy-valued functions [4]. The strongly generalized differentiability is defined by considering lateral H-derivatives. Clearly the disadvantage of strongly generalized differentiability of a function in comparison H-differentiability is that, a fuzzy differential equation has no unique solution. Thus, a generalization of the Hukuhara difference [21] and derivative for interval valued functions are presented by Stefanini and Bede. They shown that, this concept of differentiability has relationships with weakly generalized differentiability and strongly generalized differentiability [23]. Recently, a general difference is researched in [22]. Afterwards, a generalized differentiability ideas based on general difference for fuzzy-valued function recently examined in [5].

In what follows, we briefly recall the basic definitions and properties of the generalized Hukuhara derivative. We denote by $\mathbb{R}_{\mathcal{F}}$ the set of fuzzy numbers, that is, normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which are defined over the real line. For $0 < \alpha \le 1$, set $[u]^\alpha = \{x \in \mathbb{R}^n | u(x) \ge \alpha\}$, and $[u]^0 = cl\{x \in \mathbb{R}^n | u(x) > 0\}$. We represent $[u]^\alpha = [u^-(\alpha), u^+(\alpha)]$, so if $u \in \mathbb{R}_{\mathcal{F}}$, the $\alpha$-level set $[u]^\alpha$ is a closed interval for all $\alpha \in [0, 1]$. For arbitrary $u, v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$, the addition and scalar multiplication are defined by $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$, $[ku]^\alpha = k[u]^\alpha$ respectively.

A triangular fuzzy number is defined as a fuzzy set in $\mathbb{R}_{\mathcal{F}}$, that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \le b \le c$ such that $u^-(\alpha) = a + (b - a)\alpha$ and $u^+(\alpha) = c - (c - b)\alpha$ are the endpoints of $\alpha$-level sets for all $\alpha \in [0, 1]$. The support of fuzzy number $u$ is defined as follows:

$$\operatorname{supp}(u) = cl\big\{x \in \mathbb{R}^n \big| u(x) > 0\big\},$$

where $cl$ is closure of set $\{x \in \mathbb{R}^n | u(x) > 0\}$. The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup \{0\}$ as in [18]

$$D(u, v) = \sup_{\alpha\in[0,1]} d\big([u]^\alpha, [v]^\alpha\big) = \sup_{\alpha\in[0,1]} \max\big\{\big|u^-(\alpha) - v^-(\alpha)\big|, \big|u^+(\alpha) - v^+(\alpha)\big|\big\},$$