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On various approaches to normalization of interval and fuzzy weights

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Abstract

The paper deals with the problem of fuzzification of the procedure of normalization of weights. First, the existing methods for normalization of interval and fuzzy weights are reviewed. Second, we study the problem of normalization of a fuzzy vector of weights that expresses the joint possibility distribution of initial weights. We show that a correct way is to apply the extension principle proposed by Zadeh, since the result of such normalization is the fuzzy vector of normalized weights that expresses the true joint possibility distribution of normalized weights. Further, we establish some properties of this approach to normalization that are important from the point of view of real applications. Finally, since an n-tuple of non-interactive interval or fuzzy weights can be viewed as a fuzzy vector of weights of a special kind, we investigate normalization of such kind of fuzzy vectors of weights according to the extension principle. We show that from the point of view of the way of modelling uncertain normalized weights, the result of this approach can be directly compared only with the result of normalization proposed by Wang and Elhag (2006) [35]. We find out that it is not sufficient to express the result of normalization only by an n-tuple of normalized interval or fuzzy weights together with the constraint that the sum of the weights is equal to 1, since it can cause a false increase of uncertainty in the model. This fact is illustrated by an example.

Keywords: Normalization of weights; Interval weights; Fuzzy weights; Fuzzy vector; Fuzzy weighted average; Normalized fuzzy weights

1. Introduction

In multiple criteria decision making (MCDM) models, weights usually represent some kind of ordinal or cardinal information about the importance of criteria. The weights are in general expressed by nonnegative real numbers whose sum is different from zero. Particularly, if the sum of the weights is equal to 1, they are called normalized weights. Throughout the paper, we will denote the set of all n-tuples of general weights by $\mathcal{W}_n$, i.e.

$$\mathcal{W}_n := \left(\mathbb{R}_0^+\right)^n \setminus \{\mathbf{o}\},$$

where $\mathbf{o}$ denotes the zero vector, and the set of all n-tuples of normalized weights, the *n-dimensional probability simplex*, by $\mathcal{S}_n$, i.e.

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$$\mathcal{S}_n := \left\{ (v_1, \dots, v_n) \in \mathbb{R}^n \;\middle|\; v_i \geqslant 0,\ i = 1, \dots, n,\ \sum_{i=1}^{n} v_i = 1 \right\}.$$

In MCDM models, the weights of criteria are often ill-known. Their values are commonly set subjectively on the basis of experts' experiences or opinions. We can also face incomplete or missing information about the importance of criteria. Such kinds of uncertainty can be sufficiently modelled by means of tools of fuzzy sets theory.

In the literature, uncertain non-normalized weights are usually expressed (see e.g. [1,5,14,15,17,20,25,32,35]) by a tuple of non-interactive non-negative fuzzy numbers.

For describing uncertain normalized weights, it was shown in [24,25,33,35] that a special structure of interactive fuzzy numbers on [0, 1] called *a tuple of normalized fuzzy weights* has to be applied because of the normalization condition. Construction of this structure is based on the assumption that only such combinations of values of particular weights whose sum is equal to 1 are admissible. The identical structure of fuzzy numbers was in [22] called *a feasible tuple of fuzzy probabilities*. It represented a generalization of *a tuple of reachable interval probabilities* introduced e.g. in [4,21,37].

A different structure of intervals or fuzzy numbers is proposed for modelling uncertain values of normalized weights in [32]. Normalized interval and fuzzy weights are treated as intervals or fuzzy objects, not as only constrains on standard (real valued) normalized ones.

A more general approach to modelling uncertain weights was proposed in [23,28]. It was shown that uncertain n-tuples of non-normalized or normalized weights can be appropriately expressed by n-dimensional fuzzy vectors representing joint possibility distributions of the weights. In comparison to the above mentioned n-tuples of non-interactive fuzzy weights or n-tuples of normalized fuzzy weights, the fuzzy vectors significantly extend the possibilities of utilizing the vague expert information concerning the weights.

General weights often need to be normalized in MCDM models for the purpose of elimination of their dimensions. For instance, in Analytic Hierarchy Process (AHP) [30], the weight vector obtained from a pairwise comparison matrix has to be normalized so that we can aggregate the local weights in a hierarchical structure into a global weight vector. Such kind of normalization of an n-tuple of general weights to the corresponding n-tuple of normalized weights is described by the real-vector-valued function $\mathbf{n} : \mathcal{W}_n \to \mathcal{S}_n$ defined for all $(w_1, \dots, w_n) \in \mathcal{W}_n$ in the following way:

$$\mathbf{n}(w_1, \dots, w_n) := \left(\frac{w_1}{\sum_{i=1}^{n} w_i}, \dots, \frac{w_n}{\sum_{i=1}^{n} w_i} \right). \tag{1}$$

For the same reason, there is also often a necessity to normalize interval or fuzzy weights in MCDM models under uncertainty, especially in AHP with interval or fuzzy judgements (see e.g. [2,3,36,38]). Thus, the normalization function given by (1) has to be extend to the case of uncertain weights. In the literature, several methods for normalization of an n-tuple of interval or fuzzy weights were proposed up to now (see [32,35] and references therein). The methods are based on different understanding of what does the normalization mean when interval or fuzzy weights are available. Normalization of fuzzy vectors of weights has not been considered in the literature up to now, it was studied only partly in [23]. The aim of the paper is to review the existing methods for normalization of interval and fuzzy weights and to study the problem of the correct extension of the normalization procedure given by (1) to the case when uncertain weights are expressed by fuzzy vectors.

The paper is organized as follows. The existing methods to normalization of interval and fuzzy weights are discussed in Section 2. Special attention is given to the different understanding of correctness of the normalization procedure in particular methods. In Section 3, modelling of n-tuples of uncertain weights by fuzzy vectors is briefly described. The approach to normalization of weights that is based on the extension principle and that makes it possible to normalize also the fuzzy vectors of weights is studied in Section 4. Some important properties of this way of normalization of fuzzy vectors of weights are shown in Section 5. Since an n-tuple of non-interactive interval or fuzzy weights can be expressed by a fuzzy vector of weights of a special kind, its normalization according to the extension principle is studied separately in Section 6. Finally, some concluding remarks are given.

2. Existing approaches to normalization of interval and fuzzy weights

In this section, we will summarize the existing approaches to normalization of interval and fuzzy weights. We will discuss especially well-foundedness of the criteria of correctness of the normalization procedure that have been considered in the literature.