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Total orderings defined on the set of all fuzzy numbers

Wei Wang [a], Zhenyuan Wang [b]

[a] *Computer Science Department, University of California at Los Angeles, USA*
[b] *Department of Mathematics, University of Nebraska at Omaha, USA*

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Abstract

In this work, a new concept of upper dense sequence in interval $(0, 1]$ is introduced. There are infinitely many upper dense sequences in interval $(0, 1]$. Using any upper dense sequence, a new decomposition theorem for fuzzy sets is established and proved. Then, using a chosen upper dense sequence as one of the necessary reference systems, infinitely many total orderings on the set of all fuzzy numbers can be well defined. Among them, a common upper dense sequence based on the binary numbers is suggested as a natural default option. Another upper dense sequence based on the rational numbers is also suggested. Regarding real numbers as special fuzzy numbers, all of these total orderings defined by using the suggested upper dense sequences are consistent with the natural ordering of real numbers. Building total ordering on the set of all fuzzy numbers in such a way is significant for fuzzy data analysis and, therefore, may be used in decision making with fuzzy information.
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1. Introduction

Fuzzy numbers is one of the most important mathematical concepts concerning fuzziness. Ranking fuzzy numbers is an essential step in analyzing fuzzy information in optimization, data mining, decision making, and related areas. Since the introduction of the concept of fuzzy sets and fuzzy numbers in the sixties of the last century, many significant contributions have been made in ranking fuzzy numbers [1–4,7–9,14–17,19,20,22]. They have respect intuition based on some geometric characteristics (e.g., area, distance, or centroid), and can be used for various purposes. Mostly, these methods can either order a set of fuzzy numbers that are not equivalent according to some selected characteristic(s), that is, only rank fuzzy numbers but allow different fuzzy numbers to have the same rank, or order some special types of fuzzy numbers, such as the triangular (or, more generally, trapezoidal) fuzzy numbers. In addition, a total ordering for the graded numbers, which are similar to some special type of fuzzy numbers and can be identified by finitely many real-valued parameters, has been discussed [10].

Generally, based on one or more characteristics of fuzzy numbers, an equivalence relation and an opposite but transitive relation on the set of all fuzzy numbers can be defined. Using these two relations, it is easy to define a total ordering on its quotient space of the equivalence relation, but not on the set of all fuzzy numbers themselves. Up to

E-mail address: zhenyuanwang@unomaha.edu (Z. Wang).

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now, no one has reported any successful result on total orderings defined on the set of all fuzzy numbers in literature. The difficulty of defining total ordering for all fuzzy numbers is that there is no effective tool to identify an arbitrarily given fuzzy number by only finitely many real-valued parameters. In this work, by establishing a new decomposition theorem for fuzzy sets, we can use an upper dense sequence with the natural ordering of real numbers and a chosen ordering for the end points of rectangles as the necessary reference systems to overcome the above-mentioned difficulty and, then, present a way for defining various total orderings on the set of all fuzzy numbers.

This paper is arranged as follows. After the Introduction, some necessary fundamental knowledge on orderings and fuzzy numbers is reviewed in Section 2. In Section 3, we survey existing results on the total ordering defined on several sets of fuzzy numbers with some common special type. Section 4 is used to introduce and discuss the concept of upper dense sequence in interval $(0, 1]$. A new decomposition theorem for fuzzy sets is established in Section 5. Then, we define total orderings on the set of all fuzzy numbers in Section 6. Several examples showing how the total ordering can be used for ranking or ordering fuzzy numbers are presented in Section 7. Finally, conclusions are given in Section 8.

2. Orderings and fuzzy numbers

Let X be a nonempty set. Any subset of the product set $X \times X$ is called a relation, denoted by R, on X. We write aRb if and only if $(a, b) \in R$. Relation R is reflective if and only if aRa for every $a \in X$. Relation R is symmetric if and only if, for any $a, b \in X$, aRb implies bRa. Relation R is antisymmetric if and only if, for any $a, b \in X$, aRb and bRa imply $a = b$. Relation R is transitive if and only if, for any $a, b, c \in X$, aRb and bRc imply aRc. Relation R is called a partial ordering on X if it is reflective, antisymmetric, and transitive. A partial ordering R on X is called a total ordering if either aRb or bRa for any $a, b \in X$. Two total orderings R_1 and R_2 are different if and only if there exist $a, b \in X$ with $a \neq b$ such that aR_1b but bR_2a. For any given total ordered infinite set, there are infinitely many different ways to redefine a new total ordering on it. Relation R is called an equivalent relation if it is reflective, symmetric, and transitive.

Now, let $\mathbb{R} = (-\infty, \infty)$. A fuzzy subset of $\mathbb{R}$, denoted by $\tilde{e}$, is called a fuzzy number if its membership function $m_e : \mathbb{R} \to [0, 1]$ satisfies the following conditions.

(FN1) Set $\{x \mid m_e(x) \geqslant \alpha\}$, the α-cut of $\tilde{e}$ (denoted by e_α), is a closed interval for every $\alpha \in (0, 1]$.
(FN2) Set $\{x \mid m_e(x) > 0\}$, the support set of $\tilde{e}$ (denoted by $\mathrm{supp}(e)$), is bounded.

Condition (FN1) is equivalent to the following three conditions.

(FN1.1) There exists at least one real number a_0 such that $m_e(a_0) = 1$.
(FN1.2) Function m_e is nondecreasing on $(-\infty, a_0]$ and nonincreasing on $[a_0, \infty)$.
(FN1.3) Function m_e is upper semi-continuous, or say, m_e is right-continuous (i.e., $\lim_{x \to x_0+} m_e(x) = m_e(x_0)$) when $x_0 < a_0$ and is left-continuous (i.e., $\lim_{x \to x_0-} m_e(x) = m_e(x_0)$) when $x_0 > a_0$.

For any fuzzy number $\tilde{e}$, set $\{x \mid m_e(x) = 1\}$ is nonempty and is called its core (or, kernel). It is the most important factor for identifying a fuzzy number. Any real number is a special fuzzy number that can be identified by only its core, which is a singleton.

The set of all fuzzy numbers is denoted by $\mathscr{N}_F$. It is easy to define a partial ordering on set $\mathscr{N}_F$ as follows.

First, a partial ordering, denoted by $\preccurlyeq$, on the set of all closed intervals, denoted by $\mathscr{N}_I$, is defined by saying $[a, b] \preccurlyeq [c, d]$ if and only if $a \leqslant c$ and $b \leqslant d$. Then a partial ordering, still denoted by $\preccurlyeq$, on $\mathscr{N}_F$ can be well defined by saying $\tilde{e_1} \preccurlyeq \tilde{e_2}$ if and only if $(e_1)_\alpha \preccurlyeq (e_2)_\alpha$ for all $\alpha \in (0, 1]$.

The major contribution of this work is presented in Section 6, where we give total orderings defined on the set of all fuzzy numbers, $\mathscr{N}_F$.

3. Total orderings defined on some sets of fuzzy numbers with special type

Before defining total orderings on the set of all fuzzy numbers, let us recall some total orderings defined on several common proper subsets of $\mathscr{N}_F$.