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# Weak bisimulations for fuzzy automata ☆

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**Abstract**

Forward and backward bisimulations have been introduced recently by Ćirić et al. (2012) [9] as a means for modeling the equivalence between states of fuzzy automata and approximating the language-equivalence, as well as for use in the state reduction of fuzzy automata. The main aim of the present paper is to introduce two new kinds of bisimulations, weak forward and backward bisimulations, which provide better approximations of the language-equivalence than forward and backward bisimulations, and when employed in the state reduction, they provide better reductions. We give procedures for deciding whether there exist weak forward and backward simulations and bisimulations, and for computing the greatest ones, whenever they exist. Using weak bisimulations in conjunction with the concept of a uniform fuzzy relation, we determine necessary and sufficient conditions under which two fuzzy automata are weak bisimulation equivalent. We also characterize uniform weak backward and forward bisimulations between two fuzzy automata in terms of isomorphisms between their Nerode and reverse Nerode automata.


*Keywords:* Fuzzy automata; Fuzzy relation inequalities; Simulation; Bisimulation; State reduction; Equivalence of automata; Weak bisimulations; Uniform fuzzy relations; Fuzzy equivalence relations; Complete residuated lattices

## 1. Introduction

From the very beginning of the theory of fuzzy sets, fuzzy automata and languages were studied as a means for bridging the gap between the precision of computer languages and vagueness and imprecision, which are frequently encountered in the study of natural languages. Study of fuzzy automata and languages was initiated in 1960s by Santos [43,44,46], Wee [52], Wee and Fu [53], and Lee and Zadeh [30]. From late 1960s until early 2000s mainly fuzzy automata and languages with membership values in the Gödel structure have been considered (cf., e.g., [16, 17,34]). The idea of studying fuzzy automata with membership values in some structured abstract set comes back to Wechler [51], and in recent years researcher's attention has been aimed mostly to fuzzy automata with membership values in complete residuated lattices, lattice-ordered monoids, and other kinds of lattices. Fuzzy automata taking membership values in a complete residuated lattice were first studied by Qiu in [37,38], where some basic concepts were discussed, and later, Qiu and his coworkers have carried out extensive research of these fuzzy automata (cf. [39, 41,54–58]). From a different point of view, fuzzy automata with membership values in a complete residuated lattice were studied by Ignjatović, Ćirić and their coworkers in [9–12,21,23–25,47,48]. During the decades, fuzzy automata

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and languages have got a wide field of applications, including lexical analysis, description of natural and programming languages, learning systems, control systems, neural networks, knowledge representation, clinical monitoring, pattern recognition, error correction, databases, discrete event systems [40], and many other areas.

One of the most important problems of automata theory is to determine whether two given automata are language-equivalent, i.e., to determine whether they recognize the same language. For deterministic finite automata the equivalence problem is solvable in polynomial time, but for nondeterministic and fuzzy finite automata it is computationally hard (PSPACE-complete) [18,49,50]. Another important issue in the automata theory is to express the language-equivalence of two automata as a relation between their states, if such relationship exists, or find some kind of relations between states which would approximate the language-equivalence. The language-equivalence of two deterministic automata can be expressed in terms of relationships between their states, but in the case of nondeterministic and fuzzy automata the problem is more complicated, and we can only examine various approximations of the language-equivalence. The problem of equivalence of fuzzy automata was first studied by Santos in [44,45], where he considered equivalence of finite max–min fuzzy automata and max-product fuzzy automata. The equivalence problem for fuzzy automata with membership degree in a bounded chain was studied by Peeva [36], where she considered various special equivalence problems and provided their algorithmic decidability. Later, equivalence of fuzzy automata over complete residuated lattice was discussed by Xing et al. in [58].

A widely-used notion of "equivalence" between states of automata is that of bisimulation. Bisimulations were introduced, in concurrency theory, by Milner [33] and Park [35], and they have been very successfully exploited to model equivalence between various systems, as well as to reduce the number of states of these systems. Roughly, at the same time, bisimulations have been discovered in some areas of mathematics, e.g., in set theory and modal logic. Nowadays, they are widely employed in the computer science, particularly in object-oriented languages, functional languages, verification tools, data types, domains, databases, program analysis, etc. For more information of bisimulations we refer to [1,4,8,9,32].

The most common structures on which bisimulations have been studied are labeled transition systems, tree automata, weighted automata, etc. [12,19,20,33]. Recently, bisimulations have been also studied in the setting of fuzzy automata and fuzzy transition systems [5,6,9,10]. In particular, the paper [9] initiated the study of two types of simulations – forward and backward simulations, as well as of four types of bisimulations – forward, backward, backward–forward and forward–backward bisimulations. Efficient algorithms for deciding whether there are simulations and bisimulations of these types, and for computing the greatest ones, whenever they exist, were provided in [10]. It is worth noting that they are based on the computing the greatest solutions to particular systems of fuzzy relation inequalities and equations.

As we have already pointed out, forward and backward bisimulations have been introduced as a means for modeling the equivalence between states of fuzzy automata and approximating the language-equivalence, as well as for use in the state reduction of fuzzy automata. The main aim of this paper is to introduce two new kinds of bisimulations, weak forward and backward bisimulations, which provide better approximations of the language-equivalence than forward and backward bisimulations, and when employed in the state reduction, they provide better reduc tions. Our main results are the following. We give procedures for deciding whether there exist weak forward and backward simulations and bisimulations, and for computing the greatest ones, whenever they exist (Theorems 3.7 and 3.8). Then weak bisimulations are studied in conjunction with the concept of a uniform fuzzy relation, which has been introduced in [7], and further developed in [9]. In particular, we show that two fuzzy automata $\mathcal{A}$ and $\mathcal{B}$ are weak forward bisimulation equivalent, i.e., there is a uniform weak forward bisimulation between them, if and only if there is a weak forward isomorphism between the factor fuzzy automata with respect to the greatest weak forward bisimulation equivalences on $\mathcal{A}$ and $\mathcal{B}$ (Theorem 5.2). An analogous theorem can be proved for weak backward bisimulation equivalent fuzzy automata. Also, we characterize uniform weak backward and forward bisimulations between two fuzzy automata in terms of isomorphisms between their Nerode and reverse Nerode automata (Theorem 4.12).

The paper is organized as follows. In Section 2 we give definitions of basic notions and notation concerning fuzzy sets, relations and fuzzy automata, and we recall definitions of uniform fuzzy relations and bisimulations between fuzzy automata and some basic results concerning them. In Section 3 we introduce weak forward and backward simulations and bisimulations and we discuss their fundamental properties. Then in Section 4 we study uniform weak forward bisimulations, and in Section 5 we discuss the weak forward bisimulation equivalence between fuzzy automata.