Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 249 (2014) 114–127

FUZZY sets and systems

www.elsevier.com/locate/fss

The notion of roughness of a fuzzy set

Inés Couso *, Laura Garrido, Luciano Sánchez

University of Oviedo, Spain

Received 5 November 2013; received in revised form 20 January 2014; accepted 14 February 2014

Available online 26 February 2014

Abstract

The roughness of a set (according to the notion introduced by Pawlak in 1991) can be regarded as the MZ-distance between its upper and the lower approximations. With this idea in mind, we have generalized Pawlak's definition, by replacing the MZ-distance by a general "distance" measure. We also generalize the notion of roughness of fuzzy sets introduced by Huyhn and Nakamori in 2005.

Keywords: Rough set; Fuzzy set; Dissimilarity measure; Divergence measure; Roughness

1. Introduction

Rough set theory, proposed by Pawlak in 1982 [14], was motivated by practical needs to represent and process indiscernibility of individuals. According to [16], the rough set philosophy is founded on the assumption that with every object of the universe of discourse we associate some information. For example, if objects are patients suffering from a certain disease, symptoms of the disease form information about patients. Objects characterized by the same information are indiscernible (similar) in view of the available information about them, and they are formally linked by means of an equivalence relation R. Every equivalence class is regarded as a basic granule or "atom" of knowledge about the universe. Any union of some elementary sets is referred to as "precise", "exact" or "definable" set – otherwise the set is "rough", and those elements that cannot be "seen" through the available information are called "boundary-line" elements. Therefore, rough sets are those with non-empty boundary region. We can only determine a pair of lower–upper approximations of them. The lower approximation consists of all objects which surely belong to the set and the upper approximation contains all objects which possibly belong to it, according to the granular available information. In a finite setting, Pawlak [15] introduced the notion of *accuracy* of a set X with respect to the equivalence relation R in order to capture the degree of completeness of our knowledge (determined by R) about X. He denoted it $\alpha_R(X)$ and defined it as the quotient between the cardinals of its lower and upper approximations. Obviously, $0 \leqslant \alpha_R(X) \leqslant 1$ and $\alpha_R(X) = 1$ indicates that X is an exact or precise set. Otherwise, the borderline region of

* Corresponding author. Tel.: +34 985181906; fax: +34 985181902.
E-mail addresses: couso@uniovi.es (I. Couso), garridolaura@uniovi.es (L. Garrido), luciano@uniovi.es (L. Sánchez).

http://dx.doi.org/10.1016/j.fss.2014.02.013
0165-0114/

X is non-empty and X is a rough set. Pawlak also introduced the opposite notion of *R-roughness* or simply *roughness* of a set X, in order to quantify the degree of incompleteness of knowledge about X provided by R. He defined it as $\rho_R(X) = 1 - \alpha_R(X)$. In [20], Yao mentioned that Pawlak's roughness measure can be expressed as the MZ-distance between the lower and the upper approximations of X with respect to the relation R. According to this, and taking into account the extensive literature about "distance", "dissimilarity" and "divergence" measures in (fuzzy) set theory (see [4], for instance) we will propose a generalization of Pawlak's notion, and study the properties of this general definition. It will be applicable to general families of sets, not necessarily finite. Furthermore, we will also propose a general definition of the notion of roughness of a fuzzy sets, that will contain, as a particular case, the recent proposal by Huynh and Nakamori [10].

2. Preliminaries

In this section, we will provide the necessary background on measures of dissimilarity and rough sets.

2.1. Measures of dissimilarity, divergence and distance

In this subsection, we will survey some remarkable axiomatic definitions from the literature regarding the notions of "inequality" or "difference" between fuzzy sets, and we recall their formal relations. In order to list all these definitions in a compact way, we will provide two separate lists of properties: the properties included in the first list are general axioms that may be required in any kind of comparison measure between fuzzy sets. The second list is related to the notion of "difference" or "inequality". We will follow the nomenclature introduced in [4]. As a convention, the symbols "$-$" and "$+$" will be understood as "weaker than" and "stronger than", respectively. We will denote the universe by U and the family of fuzzy subsets by $\mathcal{F}(U)$. The membership function of an arbitrary subset $\tilde{A} \in \mathcal{F}(U)$ will be denoted by $\mu_{\tilde{A}}$. $\wp(U)$ with denote the family of crisp subsets of U. We will denote by $A \cap B$ and $A \cup B$, respectively, the intersection and the union of A and B according to the minimum T-norm and the maximum T-conorm, i.e., the fuzzy sets whose membership functions are respectively expressed as $\mu_{A\cap B} = \min\{\mu_A, \mu_B\}$ and $\mu_{A\cup B} = \max\{\mu_A, \mu_B\}$. A^c with denote the complement of A, with the membership function $\mu_{A^c} = 1 - \mu_A$. $A \setminus B$ will denote the difference between A and B. According to [2], it will be assumed to generalize the classical notion of difference between two crisp sets and furthermore satisfy the following properties:

Diff1. If $A \subseteq B$ then $A \setminus B = \emptyset$.
Diff2. $B \setminus A$ is monotone with respect to B ($B \subseteq B'$ entails $B \setminus A \subseteq B' \setminus A$).

Let us now introduce the three aforementioned lists of axioms.

- General properties.
  - G1. – $0 \leqslant m(\tilde{A}, \tilde{B}) \leqslant 1, \forall \tilde{A}, \tilde{B} \in \mathcal{F}(U)$.
  - G2. – $m(\tilde{A}, \tilde{B}) = m(\tilde{B}, \tilde{A}), \forall \tilde{A}, \tilde{B} \in \mathcal{F}(U)$.
  - G3. – U is finite and there exists $h : [0, 1] \times [0, 1] \to \mathbb{R}$ such that $m(\tilde{A}, \tilde{B}) = \sum_{x\in U} h[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)]$.
  - G4. – There exists a mapping $f : \mathcal{F}(U) \times \mathcal{F}(U) \times \mathcal{F}(U) \to \mathbb{R}$ such that $m(\tilde{A}, \tilde{B}) = f(\tilde{A} \cap \tilde{B}, \tilde{A} \setminus \tilde{B}, \tilde{B} \setminus \tilde{A})$.
  - G4$^+$. – There exists a function $F_m : \mathbb{R}^3 \to \mathbb{R}$ and a fuzzy measure[1] $M : \mathcal{F}(U) \to \mathbb{R}$ such that, for all $\tilde{A}, \tilde{B} \in \mathcal{F}(U)$, $m(\tilde{A}, \tilde{B})$ can be written as follows:
$$m(\tilde{A}, \tilde{B}) = F_m\big(M(\tilde{A} \cap \tilde{B}), M(\tilde{A} \setminus \tilde{B}), M(\tilde{B} \setminus \tilde{A})\big).$$
  - G5[2]. – If $\tilde{A} \cap \tilde{B} = \emptyset$, $\tilde{A}' \cap \tilde{B}' = \emptyset$, $m(\tilde{A}, \emptyset) \leqslant m(\tilde{A}', \emptyset)$ and $m(\tilde{B}, \emptyset) \leqslant m(\tilde{B}', \emptyset)$, then $m(\tilde{A}, \tilde{B}) \leqslant m(\tilde{A}', \tilde{B}')$.
- Axioms for measures of "distance", "divergence" or "dissimilarity"

[1] Several formal definitions of this notion can be found in the literature. Here, we will refer to a monotone increasing (fuzzy)-set-function satisfying the restriction $M(\emptyset) = 0$.

[2] According to this axiom, the degree of (dis)similarity between two disjoint sets A and B should be monotone with respect to their respective (dis)similarity sizes, expressed in terms of their dissimilarities with respect to the empty set.