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On an implicit assessment of fuzzy volatility in the Black and Scholes environment

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Abstract

In this work we suggest a methodology to obtain the membership of a non-observable parameter through implicit information. To this aim we profit from the interpretation of membership functions as coherent conditional probabilities. We develop full details for the well known Black and Scholes pricing model where the membership of the volatility parameter is obtained from a sample of either asset prices or market prices for options written on that asset.

Keywords: Fuzzy membership elicitation; Implicit information; Coherent conditional probability assessments and extension; Probability–possibility transformation

1. Motivation and assumptions

Most mathematical models rely on unknown quantities (parameters) which are usually estimated through sampling techniques. One of the main issues in the application of such models is the additional problem of non-observability of some of these parameters. In this case direct sampling is not possible and it is compulsory to rely on indirect information. A typical example in financial applications is the derivation of the volatility of a risky asset. In particular, in the context of Black and Scholes model [4] (BS model, hereafter), it is assumed that the price S_t of the risky asset follows a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t, \qquad (1)$$

where σ is the volatility of the asset, t is the time index, μ is the mean asset return and W_t is a standard Wiener process. Under this dynamics assumption, a closed formula can be derived for the evaluation of European-style derivatives, e.g. the price at time t for a European Call option with strike price K and expiration time T is

$$C(S_t) = \Phi(d_1)S_t - \Phi(d_2)Ke^{-r(T-t)}, \qquad (2)$$

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with

$$d_{1,2} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r \pm \frac{\sigma^2}{2}(T-t)\right)}{\sigma\sqrt{T-t}}, \tag{3}$$

r the continuously compounded riskless interest rate, and $\Phi(\cdot)$ the cdf of the standard normal distribution.

Note that the pricing formula (2) only depends on the rate r and on the volatility parameter σ. Since r is indeed independent of the stock price, σ is the most relevant parameter in (1). As a consequence, its estimation is crucial and has inspired a huge literature on estimation procedures (see, among others, [3,30,33]). Besides, generalizations of BS model have been introduced in which σ is assumed to be random; seminal papers in this direction are [26,34,39,24].

However, even in the simple constant case, the estimation of volatility is debatable. In fact, while volatility is not directly observable, indirect information is available through associated quantities such as the price S_t of the risky asset itself as well as the price of some of its derivative claims. Pure statisticians usually base their evaluations on a sample of stock prices and estimate σ by the standard deviation of sample log-returns.

On the contrary, market practitioners mostly rely on indirect information inferred from a sample of market option prices, which they claim to be more informative about the current market beliefs than asset prices themselves. There is anyhow a general agreement that both kinds of available information can only produce *vague* statements about the value of σ.

Recently, several authors proposed to explicitly include imprecision in parameters estimation by grading their admissibility through membership functions. Within the BS environment, in [40] it is assumed that the stock price at time t as well as the parameters r and σ are triangular fuzzy-numbers. This assumption is relaxed in [23] where other shapes for the memberships of the fuzzy values are adopted. By applying different methodologies both contributions derive a fuzzy price for Call and Put options with crisp maturities and strike prices. We stress that the support, the core and the shape of fuzzy variables in the applications suggested therein are assumed as known (or preliminarily assigned). In [9] the authors apply BS model to price real options; they assume both the stock price and the strike of the option to be modelled as trapezoidal fuzzy numbers $\widetilde{S}_t$ and $\widetilde{K}$. The Call option price is obtained as a fuzzy number by applying a hybrid version of formula (2), where the value in (3) is computed by replacing S_t and K with the possibilistic mean value of their fuzzified version, while the volatility parameter σ is replaced by the square-root of the possibilistic variance of the fuzzy stock price $\widetilde{S}_t$. Again, the support and the shape of fuzzy values $\widetilde{S}_t$ and $\widetilde{K}$ are initially assigned. A different approach is adopted in [36] where the authors assume that volatility is described through a fuzzy number $\widetilde{\sigma}$. They make use of the Heston stochastic volatility model [24], where the probability density function for the instantaneous variance is explicitly given. The authors profit from this density function and from a probability/possibility transformation in order to obtain the membership function for the fuzzy volatility $\widetilde{\sigma}$. Once again, in their example, parameters for the known distribution are given values. Similar remarks apply to other fuzzy generalizations of the BS environment (see, among others, [25,41]).

In this paper, we agree on introducing vagueness-imprecision for the volatility parameter in BS model; our contribution is to propose a methodology for a proper elicitation of $\widetilde{\sigma}$ which goes further the pre-assignment adopted in the quoted papers. This is done by using indirect information obtained from either asset or derivative prices. In addition, contrarily to what commonly done in the literature, the volatility parameter and its estimate are treated as two distinct entities. Nevertheless, we are able to infer on σ by observing the behavior of each selected estimator $\widehat{\theta}$ through an intermediate *pseudo*-membership which permits the flow of information from the estimator $\widehat{\theta}$ to the parameter σ.

In order to elicit such *pseudo*-memberships we profit from the interpretation of membership functions as coherent conditional probabilities provided in [11,14,15] and from their possible extensions. In fact, coherent conditional probability can be looked at as a general non-additive "uncertainty measure" $\mu(\cdot) = P(E|\cdot)$ of the conditioning events. This gives rise to a clear, precise and mathematical frame, which allows to define fuzzy subsets and to bound memberships of other fuzzy sets. Moreover, this interpretation permits the use of Bayesian methodologies to get a proper membership function by joining different pieces of probabilistic information.

Of course, our approach could be straightforwardly extended to any other field where a parametric model is involved and where the evaluation of an unknown and unobservable parameter is affected both from the available sources of indirect information and from sample variability. Potential applications could be in engineering (e.g. [6]) or in models for climate changes (see, among others, [29]).