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ScienceDirect

Fuzzy Sets and Systems 246 (2014) 78–90

FUZZY sets and systems

www.elsevier.com/locate/fss

Traces and property indicators of fuzzy relations ☆

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Received 13 February 2013; received in revised form 4 December 2013; accepted 23 January 2014

Available online 28 January 2014

Abstract

In this paper, we focus on the characterizations of various property indicators of fuzzy relations by means of left and right traces. The characterized indicators include those of reflexivity, T-asymmetry, S-completeness, T-transitivity, negative S-transitivity, T-S-semitransitivity and T-S-Ferrers property. The investigation can be regarded as an extension of and complement to the work done by Fodor on characterizing these properties in terms of traces.

Keywords: Fuzzy relations; Left and right traces; T-asymmetry; T-S-semitransitivity; T-S-Ferrers property

1. Introduction

Since the notion of a fuzzy (binary) relation was introduced by Zadeh [21], a lot of properties of fuzzy relations have been put forward for the requirement of various applications such as cluster analysis, ordering of fuzzy quantities, fuzzy choice functions (see, e.g., [3,15,17,20]). Particularly, in fuzzy decision-making, fuzzy relations are frequently employed to model various fuzzy preferences between alternatives. For this purpose, some properties such as T-asymmetry, strong S-completeness, T-transitivity, T-S-semitransitivity, T-S-Ferrers property play a central role in the formulation of particular fuzzy preference structures (see, e.g. [4,7–9,14]).

As is well known, whether a relation satisfies a property or not is completely deterministic. However, it is often difficult for a relation to satisfy some certain property, particularly in the fuzzy case. For example, for a fuzzy relation R on A, $T(R(a,b), R(b,c)) \leqslant R(a,c)$ must be satisfied for all a, b, c in A if R is T-transitive, where T is a t-norm. Sometimes, the difference $|T(R(a,b), R(b,c)) - R(a,c)|$ might be very small although $T(R(a,b), R(b,c)) \leqslant R(a,c)$ is not valid for some a, b, c in A. In this case, it is natural to develop an indicator to measure the degree to which T-transitivity is satisfied. As a matter of fact, some property indicators have been proposed by some researchers in order to express the degrees to which a fuzzy relation satisfies these properties. For example, Bělohlávek [5] suggested several indicators to measure the reflexivity, irreflexivity, T-asymmetry and T-transitivity degrees of a fuzzy relation.

☆ Research Project Supported by Shanxi Scholarship Council of China (2013052) and the Natural Science Foundation for Young Scholars of China (Grant No. 41101507).
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0165-0114/$ – see front matter
http://dx.doi.org/10.1016/j.fss.2014.01.008

In [15], strong totality and strong completeness degrees were put forward. Afterwards, more indicators were proposed and their relationships were further studied in [6,19].

The left and right traces of fuzzy relations were introduced by Fodor [11] in 1992, and further properties of traces were given by Fodor and Roubens in [12]. The left trace R^l and right trace R^r of a fuzzy relation R are defined by the R-implication generated from a t-norm T. If T is left continuous, both R^l and R^r are reflexive and T-transitive [12,14] and thus a T-preorder. Therefore, they behave well and are two important fuzzy relations in fuzzy decision-making analysis. Moreover, various properties of fuzzy relations can be characterized by means of their traces [11]. These properties include reflexivity, T-asymmetry, T-transitivity and T-S-Ferrers property, etc. Hence, traces are useful technical tools and play an important role in the study of fuzzy relations. Now the question is: Can property indicators of fuzzy relations still be characterized by means of traces? This paper aims to answer this question. As a continuation of research on property characterizations, we deal with the characterization issue of various property indicators of fuzzy relations by means of their left and right traces. The involved indicators include those of reflexivity, irreflexivity, T-asymmetry, S-completeness, T-transitivity, negative S-transitivity, T-S-semitransitivity and T-S-Ferrers property. The investigation can be regarded as an extension of and complement to the results obtained by Fodor on characterizing these properties in terms of traces [11].

The rest of the paper is organized as follows. In Section 2, we introduce some basic definitions, notations and results related to fuzzy logic connectives and fuzzy relations. In Section 3, we present our results on the characterizations of property indicators of fuzzy relations in terms of traces. Finally, we make some concluding remarks in Section 4.

2. Preliminaries

In this section, we recall some basic notions, notations and results related to fuzzy logical connectives and fuzzy relations. For the details, the reader may refer to [1,14,16].

2.1. Fuzzy logic connectives

Definition 2.1. (See [1,14].) If a function $n : [0, 1] \to [0, 1]$ is decreasing and satisfies $n(0) = 1$, $n(1) = 0$, then n is called a negation. A negation n is called non-filling if $n(x) = 1$ implies $x = 0$. A negation n is called strict if n is strictly decreasing and continuous. A strict negation n is strong if it satisfies involution: $n(n(x)) = x$ for all $x \in [0, 1]$.

An example of strong negation is $n(x) = 1 - x(x \in [0, 1])$, which is called the standard negation.

Definition 2.2. (See [14].) If $\varphi : [a, b] \to [a, b]$ is a continuous, strictly increasing function with boundary conditions $\varphi(a) = a$, $\varphi(b) = b$, then φ is called an automorphism of the interval $[a, b]$.

It is known that n is a strong negation iff there exists an automorphism φ of the unit interval such that $n(x) = \varphi^{-1}(1 - \varphi(x))$, where φ is called a generator of n [14]. We will denote the strong negation with generator φ by N_φ.

Definition 2.3. (See [16].) A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if it satisfies the following:

(1) Symmetry: $T(x, y) = T(y, x)$ for all $x, y \in [0, 1]$;
(2) Monotonicity: $T(x_1, y_1) \leqslant T(x_2, y_2)$ whenever $x_1 \leqslant x_2$, $y_1 \leqslant y_2$;
(3) Associativity: $T(T(x, y), z) = T(x, T(y, z))$ for all $x, y, z \in [0, 1]$;
(4) Boundary condition: $T(1, x) = x$ for all $x \in [0, 1]$.

A t-norm T is called left continuous if it is left continuous for each variable.

For example, $W(x, y) = \max(x + y - 1, 0)$ is a t-norm which is called the Łukasiewicz t-norm.

Let T be a t-norm, φ an automorphism of $[0, 1]$. Then T_φ defined by $T_\varphi(x, y) = \varphi^{-1}(T(\varphi(x), \varphi(y)))$ for all $x, y \in [0, 1]$ is also a t-norm, which is called the φ-transform of T [14]. For example, the φ-transform of the Łukasiewicz t-norm is

$$W_\varphi(x, y) = \varphi^{-1}\big(W\big(\varphi(x), \varphi(y)\big)\big) = \varphi^{-1}\big(\max\big(\varphi(x) + \varphi(y) - 1, 0\big)\big).$$