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A note on a deductive scheme of Dummett in classical and fuzzy logics☆

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Abstract

The main subject of logic is the study of the processes of deductive reasoning and, in particular, the validity of the so-called 'schemes' of such type of reasoning. In this paper, a first and partial study of the validity of the following deductive scheme is presented,

$$\begin{array}{r} \text{If } A, \text{ then } B \\ \text{If not}-A, \text{ then } B \\ B. \end{array}$$

This study is done assuming that the statements A and B are representable in Boolean algebras, De Morgan algebras, orthomodular lattices, or in standard algebras of fuzzy sets, and with the conditional statements 'If/then' translated into several conditional operators by taking always into account the consequence operator.

Keywords: Schemes of deductive reasoning; Dummett's scheme; Fuzzy logic; Classical logic

1. Introduction

Logic has devoted an almost exclusive attention to deductive reasoning, and for this reason it is frequently defined as a pair composed by an algebraic structure and a consequence operator between some of its subsets. The concept of consequence could be enlarged to the concept of conjecture in order to extend deduction to ordinary, everyday, or commonsense reasoning, represented by conjectural reasoning [9,13]. A common root of both types of reasonings is that a conclusion is obtained from some premises. It is interesting to find deductive schemes allowing to obtain as conclusion classical laws, for instance excluded middle law.

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In [5], Michael Dummett states that

The recognition of the law of excluded middle as valid hangs together with the admission of certain forms of inference as valid, in particular, the dilemma or argument by cases:

$$\text{“If } A,\text{ then } B\text{”, “If not-}A,\text{ then } B\text{”: Therefore, “}B\text{”.} \tag{1}$$

If scheme (1) holds, taking B=‘A or not-A’, makes that the excluded-middle principle ‘A or not-A’ (presented in its classical form [11]) follows deductively from the two premises.

A way towards checking the validity of the deductive scheme (1) can be done by a representation of the premises in an algebraic structure suitable for the corresponding problem, and through deductions allowed by the laws of such structure. After specifying a consequence operator C (in the sense of [13]), once statements ‘If x, then y’ are represented by means of some conditional operator $\rightarrow$, and the elements A and B by a and b, respectively, it should be proved that $b \in C\ (\{a \rightarrow b, a' \rightarrow b\})$, that is b is a consequence of $a \rightarrow b$ and $a' \rightarrow b$.

Notice that the operator $\rightarrow$ is intended for doing forward inferences, and thus is usually understood as verifying the *Modus Ponens* (MP) inequality $x \cdot (x \rightarrow y) \leq y$, for all x and y.

From the preliminary study presented in this paper, it follows that in the classical case (1) is deductively valid in Boolean algebras with the material conditional. The scheme is also studied in De Morgan algebras and its validity is shown in the case of the conjunctive conditional. For the case of orthomodular lattices some counter examples of the verification of the scheme are shown. Finally, in the standard algebras of fuzzy sets, (1) only holds with Mamdani–Larsen's conditionals and with some Q (Quantum) and D (Dishkant) operators (see Section 4).

2. The classical case viewed as a deductive process

As it is typical, in the case of classical reasoning in a Boolean algebra and with the so-called material conditional $x \rightarrow y = x' + y$, from a consistent set of *premises* $P = \{p_1, \ldots, p_n\}$, that is, with $p_\wedge = p_1 \ldots p_n \neq 0$ to avoid the existence of contradictory premises, ‘deduction’ can be formally represented by the consequence operator [9]:

$$C_\wedge(P) = \{q;\ p_\wedge \leq q\}.$$

With $P_0 = \{a \rightarrow b, a' \rightarrow b\}$ and provided $(a \rightarrow b) \cdot (a' \rightarrow b) = (a' + b) \cdot (a + b) = a \cdot a' + b = 0 + b = b \neq 0$, it is $(a \rightarrow b) \cdot (a' \rightarrow b) \leq b$, and $b \in C_\wedge(P_0)$. That is, b *follows deductively* from $a \rightarrow b$ and $a' \rightarrow b$ (scheme (1)).

Notes

(a) Conjunction (*and*), disjunction (*or*), and negation (*not*), are represented by $\cdot$, + and $'$, respectively. In addition, 0 is the minimum of the lattice and 1 is its maximum [2]

(b) If passing from P_0 to $P_0 \cup \{c_1, \ldots, c_n\}$, with new premises $c_1, \ldots, c_n$ such that $(a \rightarrow b) \cdot (a' \rightarrow b) \cdot c_1 \ldots c_n \neq 0$, $P_0 \subset P_0 \cup \{c_1, \ldots, c_n\}$ implies $C_\wedge(P_0) \subset C_\wedge(P_0 \cup \{c_1 \ldots c_n\})$. Hence, since $b \in C_\wedge(P_0)$ implies $b \in C_\wedge(P_0 \cup \{c_1 \ldots c_n\})$, b also follows deductively from $P_0 \cup \{c_1 \ldots c_n\}$.

(c) In any distributive lattice with complement and De Morgan laws, it is $(a' + b) \cdot (a + b) = a \cdot a' + b = (a + a')' + b = (a + a') \rightarrow b$, independently that the operator $x \rightarrow y = x' + y$ is or is not a *conditional*, that is, satisfies the *Modus Ponens* inequality $x \cdot (x \rightarrow y) \leq y$. In particular, in Boolean algebras

$$(a \rightarrow b) \cdot (a' \rightarrow b) = (a + a') \rightarrow b$$

implies that the verification of the scheme (1) is equivalent to the verification of ‘If A or not-A, then B’, since $a + a' = 1$.

(d) Given a *consistent* set of premises P (that is, such that $p_\wedge > 0$), it can be said that an element x in the algebra of representation is a *conjecture* of P, if $x' \notin C_\wedge(P)$ [13], that is, if its negation is not deducible from P under $C_\wedge$. Denoting by $Conj_\wedge(P)$ the set $\{x; x' \notin C_\wedge(P)\}$ of *conjectures* of P with respect to the consequence operator $C_\wedge$, it is enough that $C_\wedge$ verifies ‘$x \in C_\wedge(P) \Rightarrow x' \notin C_\wedge$’, to be sure that logical consequences are a particular case of conjectures. Thus,

$$Conj_\wedge(P) - C_\wedge(P) = \{x \in Conj_\wedge(P);\ 0 < x < p_\wedge\} \cup \{x \in Conj_\wedge(P);\ p_\wedge NCx\},$$