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# Matrix representation of meet-irreducible discrete copulas

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**Abstract**

Following Aguiló–Suñer–Torrens (2008), Kolesárová–Mesiar–Mordelová–Sempi (2006) and Mayor–Suñer–Torrens (2005), we continue to develop a theory of matrix representation for discrete copulas. To be more precise, we give characterizations of meet-irreducible discrete copulas from an order-theoretical aspect: we show that the set of all irreducible discrete copulas is a lattice in analogy with Nelsen and Úbeda-Flores (2005). Moreover, we clarify its lattice structure related to Kendall's $\tau$ and Spearman's $\rho$ borrowing ideas from Coxeter groups.


*Keywords:* Discrete copulas; Lattice; Permutation matrices; Alternating sign matrices; Kendall; Spearman; Coxeter group

## 1. Introduction

**Copulas.** The theory of copulas has been of fundamental importance in probability and statistics. It dates back to Sklar's Theorem (1959) [20]:

**Fact 1.1.** Let $X, Y$ be two random variables with marginal distribution functions $F$ and $G$. Then there exists a copula $C : [0, 1]^2 \to [0, 1]$ such that we can express the joint distribution $H$ of $X$ and $Y$ as $H(x, y) = C(F(x), G(y))$.

Since then, we have continued to develop this theory extensively. Quasi-copulas, a more general concept, recently appeared in Alsina–Nelsen–Schweizer (1993) [4]. These days this idea has wide applications in other areas such as Fuzzy logic and Quantitative finance.

**Motivation.** In this article, we focus on a certain class of copulas, *discrete copulas*. Why do we study this class? Here we list some results to see its importance:

- It has a lattice structure. This is analogous to Nelsen and Úbeda-Flores [14] that all non-discrete copulas have a lattice structure.
- [16, Theorem 2.4] Every quasi-copula is a certain limit of discrete ones.

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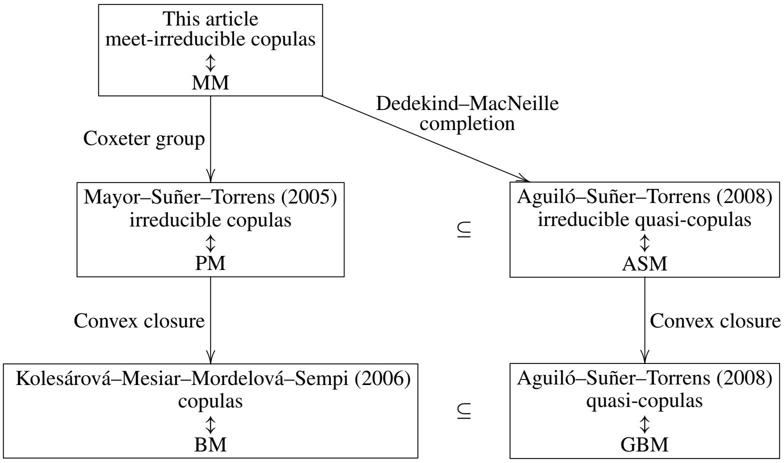


Fig. 1. Recent developments on discrete quasi-copulas.

- Discrete quasi-copulas contain rich mathematical structures of not only a matrix but also a group, a lattice and a vector space. In particular, a certain partial order fits into a framework of the classic topic on Kendall's $\tau$ and Spearman's $\rho$ such as Daniels [5], Durbin–Stuart [6], Kruskal [10], Lehmann [11] and Okamoto–Yanagimoto [15].
- We can make the best use of matrices; as in the title, our study continues recent (2000s) developments on matrix representation of discrete copulas.

In mathematics, it is a common idea to understand general objects in terms of "smaller" ones; for example, each natural number is a product of prime numbers; each element in a vector space is a linear combination of vectors in a basis. Furthermore, such expressions are often unique.

**Question 1.2.** What about discrete copulas? Is this sort of argument possible?

The answer is yes. As mentioned above, discrete copulas form a lattice. We then come to the fundamental fact in the lattice theory: in a finite distributive lattice, each non-maximal element is the meet of meet-irreducible elements. However, we could not find any references on discrete copulas from this aspect in spite of its importance.

**State of the art (Matrix representation).** Fig. 1 shows some state of the art on matrix representation of discrete copulas; there, we see the correspondences between five kinds of discrete quasi-copulas and square matrices: MM means meet-irreducible matrices, PM permutation matrices, ASM alternating sign matrices, BM bistochastic matrices and GBM generalized bistochastic matrices; see also [2] for non-square variants. Although we deal with only the first three classes, it is easy to write down several consequences for BM and GBM; details will appear in a subsequent publication. Our specific goal is to give explicit descriptions of *meet-irreducible copulas* with matrix representations. For this purpose, we "borrow" some ideas from Coxeter groups and Bruhat order.

*Coxeter groups* play a significant role in algebra, combinatorics and geometry (initiated by H.S.M. Coxeter and afterward developed by Bourbaki around 1960). Symmetric groups are indeed type A Coxeter groups equipped with a certain partial order, called *Bruhat order*. The key idea with a connection to copulas is the following:

**Fact 1.3.** Concordance order on irreducible discrete copulas on $\{0, 1, \dots, n\}$ is isomorphic to reverse Bruhat order on the symmetric group $S_n$.

This order plays a key role for studying a lattice structure of discrete copulas together with matrix representation, as we shall see.