Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 161 (2010) 2130–2149
FUZZY sets and systems
www.elsevier.com/locate/fss

# Stratified *L*-ordered convergence structures☆

Fang Jinming

*Department of Mathematics, Ocean University of China, 23 Hong Kong Road East, Qingdao 266071, PR China*

Received 3 June 2009; received in revised form 6 March 2010; accepted 5 April 2010
Available online 9 April 2010

**Abstract**

In this paper, a new kind of lattice-valued convergence structures on a universal set, called stratified $L$-ordered convergence structures, are presented by modifying the axiom for stratified $L$-generalized convergence structures in the fuzzy setting so as to make use of the intrinsic fuzzy inclusion order on the fuzzy power set. The category of stratified $L$-ordered convergence spaces described here is shown to be a reflective full subcategory in the category of stratified $L$-generalized convergence spaces, and hence it is topological and Cartesian-closed. As preparation, a further investigation of stratified $L$-filters is presented from the viewpoint that latticed-valued filters should be compatible with the intrinsic fuzzy inclusion order on the fuzzy power set.


*Keywords:* $L$-filter; $L$-ordered convergence structure; Category theory; Topology

## 1. Introduction

Lowen [25] observed that the category of stratified $L$-topological spaces is not completely satisfactory from a structural point of view, in that there is no natural function space structure for the sets $C(X,Y)$ of morphisms. In classical theory, this deficiency can be overcome by considering the category of convergence spaces [10,27], which is a super-category of the category of topological spaces [28]. In the many-valued case, where the membership lattice is $L = [0, 1]$, Lowen et al. [24,25] considered fuzzy convergence spaces as a generalization of Choquent's convergence spaces [3] and the resulting category is, among other things, Cartesian-closed. The theory of these spaces was developed to a significant extent in recent years [24,25,17,18,30]. However, Jäger [19] observed that this theory relies essentially on Lowen's definition of convergence for stratified [0,1]-topological spaces [23], where prime prefilters play a crucial role. In [12], Höhle pointed out that this theory may turn out to be void in case of more general lattices, and he further suggested an analogous convergence theory should be developed based on the concept of $L$-filters [14,15]. Following this suggestion, Jäger [19,20] developed a theory of convergence based on the notion of $L$-filters in the case when $L$ is a complete Heyting algebra. The resulting category contains the category of stratified $L$-topological spaces as a reflective subcategory and has the desired structural property of Cartesian-closedness. This work may be considered as a continuation of the very interesting research on the topic of stratified lattice-valued convergence space [19,20]. To state our objectives clearly, let us recall the definitions of stratified $L$-filters and stratified $L$-generalized convergence structures [20](or stratified $L$-fuzzy convergence structures [19]) on a set $X$, where $L$ is a complete Heyting algebra.

☆ This work is supported by the Natural Science Foundation of PR China (10771200).
*E-mail address:* jinming-fang@163.com

0165-0114/$ - see front matter 
doi:10.1016/j.fss.2010.04.001

**Definition 1.1** (*Höhle [14], Höhle and Šostak [15]*). Let $L$ be a complete Heyting algebra and $X$ be a non-void set. A mapping $\mathscr{F} : L^X \to L$ is called an $L$-filter on $X$ if it satisfies the following conditions: for all $A, B \in L^X$

(F1) $\mathscr{F}(1_X) = 1, \mathscr{F}(0_X) = 0$,
(F2) $A \le B \Rightarrow \mathscr{F}(A) \le \mathscr{F}(B)$,
(F3) $\mathscr{F}(A) \wedge \mathscr{F}(B) \le \mathscr{F}(A \wedge B)$

If, in addition, an $L$-filter $\mathscr{F}$ satisfies the condition that for all $\alpha \in L$

(Fs) $\alpha \wedge \mathscr{F}(A) \le \mathscr{F}(\alpha \wedge A)$,

it is called a stratified $L$-filter on $X$. The set of all stratified $L$-filters on $X$ is denoted by $F_L^s(X)$.

**Example 1.2** (*Eklund and Gähler [6], Höhle and Šostak [15]*). Given a point $x \in X$ the mapping $[x] : L^X \to L$ defined by for all $A \in L^X$, $[x](A) := A(x)$ is a stratified $L$-filter on $X$, called it the point $L$-filter $[x]$ of $x$.

A stratified $L$-generalized convergence structure on a set $X$ is a mapping lim from $F_L^s(X)$ to $L^X$ satisfying the following conditions:

(L1) $\lim[x](x) = 1$,
(L2) $\forall \mathscr{F}, \mathscr{G} \in F_L^s(X), \mathscr{F} \le \mathscr{G} \Rightarrow \lim \mathscr{F} \le \lim \mathscr{G}$.

All stratified $L$-generalized convergence structures on a set $X$ is denoted by $\lim_{lg}(X)$.

Furthermore, let us recall that an $L$-partially ordered set is a set $P$ together with a binary mapping $P(-,-)$: $P \times P \to L$, called an $L$-partial order, such that

(1) $P(a,a) = 1$ for every $a \in P$,
(2) $(P(a,b) = P(b,a) = 1) \Rightarrow (a = b)$ for all $a, b \in P$,
(3) $P(a,b) \wedge P(b,c) \le P(a,c)$ for all $a, b, c \in P$,

denoted by $(P, P(-,-))$ or $P$ simply. This $L$-partial order is exactly Fan-Zhang's fuzzy partial order [8,34]. In [32], Yao proved that Fan-Zhang's fuzzy partial order [8,34] and Bělohlávek's $L$-order [2] are equivalent to each other, and that both are special cases of $L-E$-order in the sense of Demirci [4,5].

For a given set $X$, define a binary mapping $\mathcal{S}(-,-) : L^X \times L^X \to L$ by $\mathcal{S}(U,V) = \bigwedge_{x \in X}(U(x) \to V(x))$ for each pair $(U,V) \in L^X \times L^X$, where "$\to$" is the implication operation on a complete Heyting algebra $L$. Then $\mathcal{S}(-,-)$ is an $L$-partial order on $L^X$. For $U, V \in L^X$, $\mathcal{S}(U,V)$ can be interpreted as the degree to which $U$ is a subset of $V$. This $L$-partial order has been known in the literature for some time. It is called the subsethood degree [9] or fuzzy inclusion order [2,35] of $L$-subsets.

If we define an $L$-partial order on $F_L^s(X)$ as follows

$$\forall \mathscr{F},\mathscr{G} \in F_L^s(X), \quad \mathcal{S}_F(\mathscr{F},\mathscr{G}) = \bigwedge_{A \in L^X} (\mathscr{F}(A) \to \mathscr{G}(A)),$$

then $(F_L^s(X), \mathcal{S}_F(-,-))$ become an $L$-partially ordered set. Hence not only $L^X$ w.r.t. $\mathcal{S}(-,-)$ but also $F_L^s(X)$ w.r.t. $\mathcal{S}_F(-,-)$ are $L$-partially ordered sets. It follows that both stratified $L$-filters $\mathscr{F}$ and stratified $L$-generalized convergence structures lim are mappings between $L$-partially ordered sets, and hence they should be compatible with the $L$-partial orders considered. Now we point out that stratified $L$-filters are already compatible with the fuzzy inclusion order $S(-,-)$ of $L$-subsets, although this is not so clear from the definition of stratified $L$-filters, but unfortunately the definition of stratified $L$-generalized convergence structures is not generally considered to be compatible with the fuzzy inclusion order $\mathcal{S}_F(-,-)$ of stratified $L$-filters.

Therefore, the first objective of this work is to show why stratified $L$-filters are compatible with the fuzzy inclusion order $S(-,-)$ of $L$-subsets; in other words, we want to show that, if an $L$-filter is compatible with the fuzzy inclusion order $S(-,-)$ of $L$-subsets, then it precisely determines a stratified $L$-filter, and to present some further properties of stratified $L$-filters. The second objective of this work is to introduce a new kind of lattice-valued convergence structures, namely, stratified $L$-ordered convergence structures, by modifying the definition of stratified $L$-generalized convergence structure, such that the stratified $L$-ordered convergence structure is compatible with the fuzzy inclusion order $\mathcal{S}_F(-,-)$ of stratified $L$-filters. Finally, the category of stratified $L$-ordered convergence spaces described here is shown to be a