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Fuzzy Sets and Systems 206 (2012) 74–88

FUZZY sets and systems

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Constrained optimization problems under uncertainty with coherent lower previsions

Erik Quaeghebeur*, Keivan Shariatmadar, Gert de Cooman

Ghent University, SYSTeMS Research Group, EESA Department, Faculty of Engineering and Architecture, Technologiepark-Zwijnaarde 914, 9052 Zwijnaarde, Belgium

Received 6 January 2011; received in revised form 1 February 2012; accepted 3 February 2012
Available online 13 February 2012

Abstract

We investigate a constrained optimization problem with uncertainty about constraint parameters. Our aim is to reformulate it as a (constrained) optimization problem without uncertainty. This is done by recasting the original problem as a decision problem under uncertainty. We give results for a number of different types of uncertainty models—linear and vacuous previsions, and possibility distributions—and for two common but different optimality criteria for such decision problems—maximinity and maximality. We compare our approach with other approaches that have appeared in the literature.

Keywords: Constrained optimization; Maximinity; Maximality; Coherent lower prevision; Linear prevision; Vacuous prevision; Possibility distribution

1. Introduction

Consider the following optimization problem:

$$\begin{array}{ll} \text{maximize} & f(x) \\ \text{subject to} & x \lhd Y, \end{array}$$

where f is a bounded real-valued objective function defined on a set $\mathscr{X}$ of optimization variables x, Y is an uncertain variable taking values in a set $\mathscr{Y}$, and $\lhd$ is a relation on $\mathscr{X} \times \mathscr{Y}$.[1] Optimization problems of this type are encountered in many fields and applications. We give some practical examples in Section 7.

To make this more concrete, consider the following toy problem. Suppose we know that Y is close to the real number c, and we are looking for the largest number x such that still $x \leq Y$. This problem could be represented mathematically as follows. The linguistic information 'Y is close to the real number c' can, for example, be modeled by a possibility

* Corresponding author.
E-mail addresses: Erik.Quaeghebeur@UGent.be (E. Quaeghebeur), Keivan.Shariatmadar@UGent.be (K. Shariatmadar), Gert.deCooman@UGent.be (G. de Cooman).
URLs: http://users.UGent.be/∼equaeghe (E. Quaeghebeur), http://users.ugent.be/∼kshariat (K. Shariatmadar), http://users.UGent.be/∼gdcooma (G. de Cooman).

[1] The boundedness of f is not an essential technical requirement, but we make it here in order to allow us to work with the more standard imprecise probability models [25], which require boundedness. For discussions of imprecise probability models without this limitation, we refer to [23,24].

0165-0114/$ - see front matter
doi:10.1016/j.fss.2012.02.004

distribution π on the set $\mathbb{R}$ of real numbers with mode c,[2] and we are looking for the solution of the following optimization problem:

$$\begin{aligned} &\text{maximize} && x \\ &\text{subject to} && x \leq Y, \end{aligned}$$

which can be seen as a special case of the more general version, where f is the identity function, and $\lhd := \leq$.[3]

The main difficulty with these problems is the uncertainty that appears in the constraint: because the value of Y is uncertain, it is uncertain for which values of x the constraint $x \lhd Y$ is satisfied, and therefore the domain over which $f(x)$ has to be maximized is not well known. The optimization problem is therefore ill-posed: it is underspecified, as there is no unique way of interpreting what is meant by maximizing a function over an uncertain domain; it might also be over-specified, as it may be infeasible in some particular case.

We will address this difficulty in Section 2 by reducing the optimization problem to a well-posed decision problem from which the uncertainties present in the description of the constraint have been eliminated. We mean by this that uncertain variables no longer appear in the problem; this is achieved by appropriately replacing the uncertain variables in the original formulation by their uncertainty models. For example, a naive way of doing this, when the uncertainty model is a classical probability distribution and $\mathscr{Y} \subseteq \mathbb{R}$, would consist of replacing the uncertain variable by its expectation.

We then investigate what results can be obtained for different types of uncertainty models for the uncertain variable Y—expectation operators or probability measures in Section 3, intervals or vacuous previsions (see, e.g., [25]) in Section 4, and possibility distributions (see, e.g., [7]) in Section 5—and for two different optimality criteria for decision problems—maximinity and maximality (see, e.g., [22]). After this, in Section 6, we compare our approach with related work.

The salient feature of our approach is that we use the same methodology and mathematical theory for solving all of these problems. Indeed, probability measures, intervals and (numerical) possibility measures can all be seen as special cases of a common and unifying uncertainty modeling framework: the theory of coherent lower previsions (see, e.g., [25,15]). Since we are convinced that coherent lower previsions (and imprecise probability models in general) provide a useful interpretational framework for possibility distributions (see also our work on possibilistic lower previsions [4]), we believe this paper establishes a novel approach to possibilistic optimization. It has the distinct advantage that the results we derive have a clear and useful behavioral interpretation.

In the remainder of this Introduction we provide some further background information. In Section 1.1, we give a brief introduction to coherent lower previsions and how they (mathematically) generalize both probability and possibility distributions as uncertainty models. In Section 1.2, we do the same for decision making with coherent lower previsions. We deal with some odds and ends in Section 1.3.

1.1. Coherent lower and upper previsions

Consider an uncertain variable Y that can take values in a set $\mathscr{Y}$. The uncertainty of an agent dealing with Y is described using an uncertainty model. The uncertainty model allows the agent to draw inferences about Y and functions thereof and make decisions in problems involving Y.

The classical uncertainty model is the probability distribution; in this context Y is usually called a random variable. Working with probability distributions or probability measures is equivalent to working with expectation operators [28]. We use the terminology of de Finetti [6] and call such expectation operators linear previsions. A linear prevision is a real functional P defined on the space $\mathscr{G}(\mathscr{Y})$ of bounded real-valued functions on $\mathscr{Y}$, that satisfies the following three so-called coherence conditions:

$$\text{Positivity:} \quad P(g) \geq \inf g, \tag{P1}$$

[2] For a discussion of why this makes sense when modeling uncertainty with imprecise probability models, see [27].

[3] The technical problem that the identity function is not bounded on the reals can be overcome, albeit somewhat artificially, by letting $\mathscr{X}$ be a bounded real interval that is large enough. Alternatively, we could extend the present discussion to include imprecise probability models that allow for unbounded gain functions, as discussed in [23,24]. But this goes beyond the scope of this paper, which is intended as an exposition of ideas and first principles.