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Dual tableau for monoidal triangular norm logic MTL[☆]

Joanna Golińska-Pilarek[a,b,*], Ewa Orłowska[b]

[a] Institute of Philosophy, Warsaw University, Poland
[b] National Institute of Telecommunications, Warsaw, Poland

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Abstract

Monoidal triangular norm logic MTL is the logic of left-continuous triangular norms. In the paper we present a relational formalization of the logic MTL and then we introduce relational dual tableau that can be used for verification of validity of MTL-formulas. We prove soundness and completeness of the system.

Keywords: Fuzzy logic; Monoidal triangular norm logic; Relational logic; Dual tableau system

1. Introduction

Monoidal triangular norm logic, MTL, was introduced in [6]. From the perspective of substructural logics it is the logic of full Lambek calculus endowed with the rules of exchange and weakening, FL_{ew}, with the additional axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$ referred to as prelinearity. From the perspective of fuzzy logics it is a logic of left-continuous triangular norms, t-norms for short (see [12]).

The classical examples of t-norms which are applied to modelling logical operator of conjunction include Łukasiewicz t-norm $x \odot y = \max(0, x+y-1)$, product t-norm defined as multiplication of reals, and Gödel t-norm $x \odot y = \min(x, y)$. All those t-norms are continuous. A t-norm $\odot$ is left-continuous iff it has a residuum, $\to$, i.e., the two operators satisfy $z \leq x \to y$ iff $x \odot z \leq y$. The residua of the three t-norms mentioned above play the role of implications: Łukasiewicz implication $x \to y = \min(1, 1-x+y)$, Gougen implication $x \to y = 1$ if $x \leq y$ and y/x otherwise, and Gödel implication $x \to y = 1$ if $x \leq y$ and y otherwise, respectively.

Algebraic semantics of logic MTL is provided by the class of MTL-algebras. They are abstract counterparts to the standard structures $([0, 1], \leq, \odot, \to)$, where $\odot$ is a left-continuous t-norm and $\to$ is its residuum. A completeness of logic MTL with respect to the semantics determined by those standard structures is presented in [11]. In [18] a Kripke-style semantics for a first-order version of logic MTL is presented, defined along the lines of semantics for substructural logics presented in [22,23]. In that semantics an algebraic structure is assumed in the universes of the models.

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* Corresponding author at: Institute of Philosophy, Warsaw University, Poland. Tel.: +48 225520140.
E-mail addresses: j.golinska@uw.edu.pl (J. Golińska-Pilarek), E.Orlowska@itl.waw.pl (E. Orłowska).

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In this paper, first, we present MTL-algebras which provide an algebraic semantics of the logic MTL. Next, we present logic MTL with a purely relational semantics developed in [4,28]. Based on this semantics a relational dual tableau for the logic is developed.

Relational dual tableaux are based on Rasiowa–Sikorski diagrams for first-order logic [29]. The common language of all relational dual tableaux is a logic of relations. Relational deduction systems in the Rasiowa–Sikorski style have been constructed for a great variety of theories, e.g., intuitionistic, modal, relevant, and multiple-valued logics, as well as for temporal logic, various logics of programs, logics of rough sets, theories of spatial reasoning including region connection calculus, theories of order of magnitude reasoning, and formal concept analysis. A survey of these results is presented in [24]. Specific methodological principles of construction of dual tableaux which make possible such a broad applicability of these systems are:

- first, given a theory, a truth preserving translation is defined of the language of the theory into an appropriate relational language (most often a language of binary relations);
- second, a dual tableau is constructed for this relational language so that it provides a deduction system for the original theory.

This methodology, reflecting the paradigm “Formulas are Relations”, enables us to represent within a uniform formalism the three basic components of formal systems: syntax, semantics, and deduction apparatus. Relational approach enables us to build dual tableaux in a systematic modular way. First, deduction rules are defined for the common relational core of the theories. These rules constitute a basis of all the relational dual tableau proof systems. Next, for any particular theory specific rules are added to the basic set of rules. They reflect the semantic constraints assumed in the models of the theory. As a consequence, we need not implement each deduction system from scratch, we should only extend the basic system with a module corresponding to the specific part of a theory under consideration. Relational dual tableaux are powerful tools which can be used to solve the four major problems that may be formulated in a theory:

- proving general laws holding in the theory;
- proving that from a finite number of laws of the theory some other law follows;
- proving that a particular model of the theory obeys some laws;
- proving that some objects which the theory deals with satisfy a law.

These tasks are represented, respectively, as: the validity problem, the entailment, the model checking problem, and the satisfaction problem in the relational logic. A recent implementation of a proof system for a relational logic is available in [8]. In [7] an implementation of translation procedures from non-classical logics to a relational logic is presented. In [10,2], implementations of relational logics for order of magnitude reasoning are presented.

The paper is organized as follows. In Section 2 we present MTL-algebras which provide algebraic semantics of monoidal triangular norm logic MTL. In Section 3 we define the logic MTL, its language and semantics. Relational formalization of logic MTL is presented in Section 4. In Section 5 we present a relational dual tableau that can be used for verification of validity of MTL-formulas. We prove its soundness and completeness in Section 6. In Section 7 we discuss alternative forms of the rules of the dual tableau for MTL. Finally, comments and conclusions are given in Section 8.

2. MTL-algebras

A t-norm is a binary operation on the closed real interval $\odot : [0,1]^2 \to [0,1]$, such that for all $x, y, z \in [0,1]$ the following hold:

- $\odot$ is associative and commutative;
- 1 is the neutral element of $\odot$;
- If $x \leq y$, then $x \odot z \leq y \odot z$.

Let $(x_i)_{i \in J}$ be an indexed family of elements of [0,1]. A t-norm is said to be *left-continuous* whenever $(\sup x_i) \odot y = \sup(x_i \odot y)$ for every $y \in [0,1]$.