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Fixed points in weak non-Archimedean fuzzy metric spaces

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Abstract

Miheţ [Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces, Fuzzy Sets and Systems, 159 (2008) 739–744] proved a theorem which assures the existence of a fixed point for fuzzy ψ-contractive mappings in the framework of complete non-Archimedean fuzzy metric spaces. Motivated by this, we introduce a notion of weak non-Archimedean fuzzy metric space and prove that the weak non-Archimedean fuzzy metric induces a Hausdorff topology. We utilize this new notion to obtain some common fixed point results for a pair of generalized contractive type mappings.

Keywords: Common fixed points; Weak non-Archimedean fuzzy metric spaces; Fuzzy contractive mappings

1. Introduction and preliminaries

The concept of fuzzy metric space was introduced in different ways by some authors (see i.e. [4,11]) and further to this, the fixed point theory in this kind of spaces has been intensively studied (see [5,9,10]). Here, we underline as the notion of fuzzy metric space, introduced by Kramosil and Michalek [11] was modified by George and Veeramani [6,8] that obtained a Hausdorff topology for this class of fuzzy metric spaces. Recently, Miheţ [12] enlarged the class of fuzzy contractive mappings of Gregori and Sapena [10] and proved a fuzzy Banach contraction result for complete non-Archimedean fuzzy metric space [12, Theorem 3.16].

Now, we briefly describe our reasons for being interested in results of this kind. The applications of fixed point theorems are remarkable in different disciplines of mathematics, engineering and economics in dealing with problems arising in approximation theory, game theory and many others (see [13] and references therein). Consequently, many researchers, following the Banach contraction principle, investigated the existence of weaker contractive conditions or extended previous results under relatively weak hypotheses on the metric space. The starting point of our paper is to follow this trend by introducing, with the definition of weak non-Archimedean fuzzy metric space, a more general setting than non-Archimedean fuzzy metric space. The reader is referred to [3] for some discussion and applications on non-Archimedean metric spaces and its induced topology. For example, let X be a non-Archimedean metric space, some assumptions on X can allow to extend a group of isometries of X to the group of Mobius transformations

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on X. Additionally, this result applies when the metric space is a field, i.e. the p-adic numbers Q_p, and it is known that many metrics arise from valuations on a ring. Also for this, our result can be of interest in such areas of mathematics as algebra, geometry, group theory, functional analysis and topology.

In this paper, we present a Hausdorff topology induced by a weak non-Archimedean fuzzy metric and some properties. Then, we utilize this new notion to obtain some common fixed point results for a pair of generalized contractive type mappings. Our results substantially generalize and extend several comparable results of Miheţ [12].

For the sake of completeness, we briefly recall some basic concepts used in the following.

Definition 1 (*Kramosil and Michalek [11]*). A fuzzy metric space is a triple $(X,M,*)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,+\infty[$, satisfying the following properties:

(KM-1) $M(x, y, 0) = 0$ for all $x, y \in X$,
(KM-2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,
(KM-3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$,
(KM-4) $M(x, y, \cdot) : [0, +\infty[\to [0, 1]$ is left continuous for all $x, y \in X$,
(KM-5) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $x, y, z \in X$ and $t, s > 0$.

In the above definition, if the triangular inequality (KM-5) is replaced by the following:
(NA) $M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s)$ for all $x, y, z \in X$ and $t, s > 0$,
then the triple $(X,M,*)$ is called a non-Archimedean fuzzy metric space. It is easy to check that the triangle inequality (NA) implies (KM-5), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

Example 1. Let $X = [0, +\infty[$, $a * b \leq ab$ for every $a, b \in [0, 1]$ and d be the usual metric. Define $M(x, y, t) = e^{-d(x,y)/t}$ for all $x, y \in X$.

Then $(X,M,*)$ is a non-Archimedean fuzzy metric space. Clearly, $(X,M,*)$ is also a fuzzy metric space.

In Definition 1, if the triangular inequality (KM-5) is replaced by the following:

$$\text{(WNA)} \quad M(x, z, t) \geq \max\{M(x, y, t) * M(y, z, t/2), M(x, y, t/2) * M(y, z, t)\}$$

for all $x, y, z \in X$ and $t > 0$, then the triple $(X,M,*)$ is called a weak non-Archimedean fuzzy metric space. Obviously every non-Archimedean fuzzy metric space is itself a weak non-Archimedean fuzzy metric space.

Remark 1. Condition (WNA) does not implies that $M(x, y, \cdot)$ is nondecreasing and thus a weak non-Archimedean fuzzy metric space is not necessarily a fuzzy metric space. If $M(x, y, \cdot)$ is nondecreasing, then a weak non-Archimedean fuzzy metric space is a fuzzy metric space.

Example 2. Let $X = [0, +\infty[$, $a*b = ab$ for every $a, b \in [0, 1]$. Define $M(x, y, t)$ by: $M(x, y, 0) = 0$, $M(x, x, t) = 1$ for all $t > 0$, $M(x, y, t) = t$ for $x \neq y$ and $0 < t \leq 1$, $M(x, y, t) = t/2$ for $x \neq y$ and $1 < t \leq 2$, $M(x, y, t) = 1$ for $x \neq y$ and $t > 2$.

Then $(X, M, *)$ is a weak non-Archimedean fuzzy metric space, but it is not a fuzzy metric space.

2. Topology induced by a weak non-Archimedean fuzzy metric

In this section, using the same arguments as in [6,7], we introduce the topology induced by a weak non-Archimedean fuzzy metric and give some properties of this topology.

Definition 2. Let $(X, M, *)$ be a weak non-Archimedean fuzzy metric space. We define open ball $B(x, r, t)$ with centre $x \in X$ and radius $r \in]0, 1[$, $t > 0$ as

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$