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States on semi-divisible generalized residuated lattices reduce to states on MV-algebras

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Abstract

A semi-divisible residuated lattice is a residuated lattice L satisfying an additional condition weaker than that of divisibility. Such structures are related to mathematical fuzzy logic as well as to extended probability theory by the fact that the subset of complemented elements induces an MV-algebra. We define generalized residuated lattices by omitting commutativity of the corresponding monoidal operation and study semi-divisibility in such structures. We show that, given a good generalized residuated lattice L, the set of complemented elements of L, denoted by $MV(L)$, forms a pseudo-MV-algebra if and only if L is semi-divisible. Maximal filters on a semi-divisible generalized residuated lattice L are in one-to-one correspondence with maximal filters on $MV(L)$. We study states on semi-divisible generalized residuated lattices. Riečan states on a semi-divisible generalized residuated lattice L are determined by Riečan states on $MV(L)$. The same holds true for Bosbach states whenever L is a good divisible generalized residuated lattice. Extremal Riečan states on a semi-divisible generalized residuated lattice L are in one-to-one correspondence with maximal and semi-normal filters on L.

Keywords: Residuated lattice; Pseudo-Wajsberg algebra; Pseudo-MV-algebra; Filter; Probability theory

1. Introduction

In this paper we study an algebraic structure called *semi-divisible generalized residuated lattice*. Since terminology in this field is not yet coherent, we adopt terminology we have used since 1987 [23]: in algebraic structures to be studied here, and without prefix *generalized* or *pseudo*, the monoidal operation $\odot$ is assumed to be commutative, while adding such a prefix the commutativity of $\odot$ need not always hold. We are aware that some authors would call our generalized residuated lattices just residuated lattices (cf. [6] or [5]) or bounded $R\ell$-monoids (cf. [10]).

To explain how semi-divisible residuated lattices are related to Hajek's [13] mathematical fuzzy logic and Höhle's monoidal logic [15], we start by quoting Von Plato [28, pp. 200–201]:

The starting point of Brouwer's criticism of classical reasoning is the indirect derivation of an existential statement, which in no way recognizes the object claimed to exist. The weak point which has no convincing justification is captured

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in the principle of excluded middle. *It asserts the truth of* $A \vee \sim A$ *for any A. An equivalent formulation is the law of double negation*: $\sim\sim A \rightarrow A$. *In* [1] *Brouwer gave the constructive notion of negation*: *If* $\perp$ *is an impossible proposition* (*an* absurdity), $\sim A$ *is defined as* $A \rightarrow \perp$. *The law of double negation says that from the impossibility of the impossibility of A follows the truth of A.* $\cdots$ *One way of seeing the implausibility of the universal validity of the principle of excluded third is offered by Kolmogorov* [18]. *He interprets the propositions* $A, B, C, \cdots$ *as expressing problems. Thus*, $A\&B$ *is a problem that is solved by solving both A and B*, $A \vee B$ *is solved by solving at least one of A or B*, *and* $A \rightarrow B$ *is solved by reducing the solution of B to that of A. This gives also negation*: *A is impossible if it leads to the solution of the absurdity* $\perp$. *The problem* $A \vee \sim A$ *would be solved if we had a way*, *applicable to just any problem A*, *of solving A or showing A impossible*, *which clearly is not the case*.

Kolmogorov's great objective in the 1925 *paper* '*On the principle of excluded middle*' [17] *was to show that* '*all the finitary conclusions obtained by means of a transitive use of the principle of excluded third are correct and can be proved even without its help*'. *To show this*, *he notes that* $\sim\sim\sim A \rightarrow \sim A$ *holds constructively. Then he replaces judgments of classical logic with their* double negation translations: *All its formulas* $\cdots$ *can in fact be so transformed that the principle of double negation elimination only finds applications to negative formulas. The upshot is that for any classically provable formula*, *there is a constructively provable formula* classically *equivalent to it*.

Heyting's [14] *well-known formalization of intuitionistic logic*, *the basis of all subsequent work on the topic*, *refers the idea to this* (*Kolmogorov's*) *paper*.

Today Kolmogorov's idea is better known as *Glivenko theorem* asserting that if a propositional formula admits a classical proof, then its double negation admits an intuitionistic proof. Hence, we have a classical proof of a propositional formula if and only if we have an intuitionistic proof of its double negation. This property is not true for predicate formulas as is shown in [16]. Via *Lindenbaum theorem*, Kolmogorov's idea can be translated into the language of algebra: the subset $MV(L) = \{x^{**} : x \in L\}$ of a Heyting algebra L is a Boolean subalgebra. In [27] we showed that the subset $MV(L)$ of Hajek's BL-algebra L can be viewed as the largest subalgebra of L which is simultaneously an MV-algebra. Thus, Lukasiewicz infinite valued propositional logic (whose algebraic counterpart is an MV-algebra) is related to Hajek's basic fuzzy logic (whose algebraic counterpart is a BL-algebra) in the same way than intuitionistic logic is related to Boolean logic.

All the above mentioned algebraic structures are known to be examples of *residuated lattices*. Thus, we seeked for the most general residuated lattice whose subset of double complemented elements, called *the MV-center of L*, induces an MV-algebra. By this we mean that $\langle MV(L), \oplus_M, ^*, \mathbf{0}, \mathbf{1}\rangle$ is an MV-algebra, where $x \oplus_M y = (x^* \odot y^*)^*$, $\odot$ is the monoidal operation of L and $\mathbf{0}, \mathbf{1}$ are the bottom and top elements of L, respectively. We showed in [26] that given a residuated lattice L, the subset $MV(L)$ induces an MV-algebra if, and only if L is semi-divisible. However, the MV-center need not to be a subalgebra of L.

The corresponding propositional logic called *semi-divisible logic* is studied in [25] where it is proved that this logic, obtained by adding some axioms and an infinitary rule, can be viewed as an axiomatic extension of Höhle's monoidal Logic [15]. Moreover, it is proved in [25] that semi-divisible logic is complete with respect to complete semi-divisible residuated lattices and that the negative part of this logic in a certain sense induces Lukasiewicz logic. Thus, by their connection to Lukasiewicz logic, semi-divisible residuated lattices are closely related to mathematical fuzzy logic [13,15]. In this paper we do not; however, study non-commutative extensions of semi-divisible logic.

On the other hand, semi-divisible residuated lattices are related to probability theory in the following way. In 1986 Mundici [20] extended probability theory on MV-algebras by defining *states* on these algebras and showed the advantages of such approach in quantum logic framework (see also [21]). In 2006 Mundici [22] showed that his approach fits well to De Finetti's subjective probabilities, too (cf. [28]). After Mundici's results on probability theory on MV-algebras, the notion of state was extended to even more general residuated structures, see e.g. [10] and references thereon. However, our main probability theoretical result in [26] was that (Riečan) states, i.e. additive measures of truth on a semi-divisible residuated lattice L, are uniquely defined by states on the corresponding MV-center $MV(L)$, which is an MV-algebra. Thus, in this sense more general approaches to probability theory reduce, after all, to Mundici's approach.

The main question in this paper, which is probability theoretical by nature, is to study what happens if we omit the commutativity assumption of the monoidal operation $\odot$ of semi-divisible residuated lattice. That is, to what extent states on a semi-divisible *generalized* residuated lattices reduce to states on the corresponding $MV(L)$. It turns out that most results from commutative case hold in semi-divisible generalized residuated lattices, too. Even more will be true: (Riečan) states on semi-divisible generalized residuated lattice L reduce to states on an MV-algebra.