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On the structure of the k-additive fuzzy measures

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Abstract

In this paper we present some results concerning the vertices of the set of fuzzy measures being at most k-additive. We provide an algorithm to compute them. We give some examples of the results obtained with this algorithm and give lower bounds on the number of vertices for the $(n-1)$-additive case, proving that it grows much faster than the number of vertices of the general fuzzy measures. The results in the paper suggest that the structure of k-additive measures might be more complex than expected from their definition and, in particular, that they are more complex than general fuzzy measures.

Keywords: Fuzzy measures; k-Additive measures; Vertices

1. Introduction

Fuzzy measures [30] (also known as capacities [2] or non-additive measures [8]) are a generalization of probability distributions. More concretely, they are measures in which the additivity axiom has been relaxed to a monotonicity condition. This extension is needed in many practical situations, in which additivity is too restrictive. Thus defined, fuzzy measures, together with Choquet integral [2], have proved themselves to be a powerful tool in many different fields, as Decision Theory [11,12], Game Theory [28,10], and many others (see for example [13] for some applications of fuzzy measures in other frameworks).

However, despite all these advantages, the practical application of fuzzy measures is limited by the increasing complexity of the measure. If we consider a finite space of cardinality n, only $n-1$ values are needed in order to completely determine a probability, while $2^n - 2$ coefficients are needed to define a fuzzy measure on the same referential. This exponential growth is the actual *Achilles' heel* of fuzzy measures. With the aim of reducing this complexity several subfamilies have been defined. In these subfamilies some extra constraints are added in order to decrease the number of coefficients but, at the same time, keep the modelling capabilities of the measures in the subfamily as rich as possible. Examples of subfamilies include the λ-measures [31], the k-intolerant measures [19], the p-symmetric measures [23], the decomposable measures [9], etc.

In this paper we will focus on the subfamily of k-additive measures, introduced by Grabisch in [10]. The concept of k-additive measure generalizes that of probability, but it is not as general as fuzzy measures. Indeed, the subfamilies of

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k-additive measures for increasing values of k determine a gradation between probabilities and general fuzzy measures. Another interesting property of k-additive measures relies on the fact that the number of coefficients increases with k; thus, we can choose the value of k in terms of the desired complexity.

Let us denote by $\mathcal{FM}^k(X)$ the set of fuzzy measures on X being k'-additive for some $k' \leq k$. It can be proved that $\mathcal{FM}^k(X)$ is a convex polytope, so that it can be characterized in terms of its vertices. In this paper we deal with the problem of determining this set of vertices. Apart from the mathematical interest of this problem, determining the set of vertices of this polytope arises in the practical identification of fuzzy measures from sample data. More concretely, we have developed in [4] a procedure based on genetic algorithms (see [15] for basic properties of these algorithms) for determining the fuzzy measure that best fits a set of data. In such procedure the cross-over operator was the convex combination between individuals and, as shown in [4], it is then necessary to consider as initial population the set of vertices.

A first study about the set of vertices of $\mathcal{FM}^k(X)$ appears in [22]; in that paper, it is proved that, unexpectedly, there are vertices of $\mathcal{FM}^k(X)$, $k \geq 3$ that are not $\{0, 1\}$-valued, i.e. such that the fuzzy measure attains other values different from 0 and 1 (cf. Propositions 1 and 2 and Theorem 2). These vertices are convex combinations of $\{0,1\}$-valued measures that are not in $\mathcal{FM}^k(X)$. Another problem related to the previous one is the problem of determining the number of vertices of $\mathcal{FM}^k(X)$. For the general case, we have proved [4] that the number of vertices for the general case coincides with the n-th Dedekind number [7]. The results in this paper seem to point out that the number of vertices of $\mathcal{FM}^k(X)$ is even greater.

The paper is organized as follows: in next section we give the basic concepts and results that will be needed in the paper. In Section 3, we provide an algorithm to generate the vertices of $\mathcal{FM}^k(X)$. This algorithm allows us to compute the number of vertices of $\mathcal{FM}^k(X)$ ($k > 2$) when $|X|$ is reduced; moreover, it seems to mean that the number of vertices of $\mathcal{FM}^k(X)$ is much larger than the corresponding number of vertices of $\mathcal{FM}(X)$. This result is proved for general $n = |X|$ and $k = n-1$ in Section 5. Section 4 deals with the special case of the algorithm for $k = n-1$. We finish with the conclusions and open problems.

2. Basic concepts and previous results

Consider a finite referential set $X = \{1, \ldots, n\}$ of n elements. Let us denote by $\mathcal{P}(X)$ the set of subsets of X. Subsets of X are denoted $A, B, \ldots$ and also by $A_1, A_2, \ldots$.

Definition 1 (*Choquet [2], Denneberg [8], Sugeno [30]*). A *fuzzy measure*, *non-additive measure* or *capacity* over X is a function $\mu : \mathcal{P}(X) \to [0, 1]$ satisfying

- $\mu(\emptyset) = 0$, $\mu(X) = 1$ (boundary conditions).
- $\forall A, B \in \mathcal{P}(X)$, if $A \subseteq B$, then $\mu(A) \leq \mu(B)$ (monotonicity).

We will denote the set of all fuzzy measures over X by $\mathcal{FM}(X)$.

Definition 2. Let $\mu \in \mathcal{FM}(X)$; we define the *dual measure of* μ as the fuzzy measure $\overline{\mu}$ given by $\overline{\mu}(A) = 1 - \mu(A^c)$.

Notice that, as $\mathcal{FM}(X)$ is an intersection of semispaces and due to the boundary conditions, it is a bounded convex polyhedron. Consequently, any fuzzy measure μ can be put as a convex combination of the vertices, which are given in the following result.

Proposition 1 (*Radojevic [26]*). *The set of* $\{0,1\}$*-valued measures constitutes the set of vertices of* $\mathcal{FM}(X)$.

Despite this simple structure, the number of vertices of $\mathcal{FM}(X)$ is not simple at all. Notice that for a $\{0,1\}$-valued measure μ, there are some non-empty subsets A satisfying the following conditions:

$$\mu(A) = 1,$$

$$\mu(B) = 1, \quad \forall B \supseteq A,$$

$$\mu(C) = 0, \quad \forall C \subset A.$$