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A survey of fuzzifications of frames, the Papert–Papert–Isbell adjunction and sobriety☆

Wei Yao

Department of Mathematics, Hebei University of Science and Technology, Shijiazhuang 050018, PR China

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Abstract

This paper gives a survey of fuzzifications of frames, the Papert–Papert–Isbell adjunction between the categories of topological spaces and locales as well as the related concept of sobriety. It then establishes categorical isomorphisms between Yao-L-frames, Zhang–Liu-L-frames and L-algebras. Lastly, it studies the relationships between L-sobriety, modified L-sobriety and $\imath_L$-sobriety. The obtained results suggest that modified L-sobriety is the most fruitful notion between the above concepts of lattice-valued sobriety.

Keywords: Topology; Category; Pultr–Rodabaugh-L-frame; Zhang–Liu-L-frame; Yao-L-frame; L-algebra; L-sober; Modified L-sober; $\imath_L$-sober

1. Introduction and preliminaries

The Papert–Papert–Isbell adjunction [14,23] between topological spaces and locales provides an appropriate environment in which to develop both topology and the theory of locales. This adjunction also gives rise to the concept of sobriety, which plays an important role in Stone Representation Theorems.

Recall the Papert–Papert–Isbell adjunction $\Omega \dashv Pt$ between the categories **Top** of topological spaces and **Loc** of locales. Firstly, for a topological space (X, τ), the family τ is a frame under the order of set inclusion. Let $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ be a continuous map between topological spaces. The map $f^{\leftarrow}|_{\tau_Y} : (\tau_Y, \subseteq) \longrightarrow (\tau_X, \subseteq)$ is a **Frm**-morphism, where $f^{\leftarrow} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ is the backward powerset operator of f given by $f^{\leftarrow}(B) = \{x \in X \mid f(x) \in B\}$. Then $\Omega(f) = (f^{\leftarrow}|_{\tau_Y})^{op}$ is a **Loc**-morphism from $(\tau_X, \subseteq)$ to $(\tau_Y, \subseteq)$. It follows that Ω is a functor from **Top** to **Loc**. Secondly, let A be a locale and let $Pt(A)$ be the set of all **Frm**-morphisms from A to $\mathbf{2} = \{\bot_{\mathbf{2}}, \top_{\mathbf{2}}\}$. Define $\Phi : A \longrightarrow \mathbf{2}^{Pt(A)}$ by $\Phi(a) = \{p \in Pt(A) \mid p(a) = \top_{\mathbf{2}}\}$. Then $\Phi^{\rightarrow}(A)$ is a topology on $Pt(A)$, the corresponding topological space still denoted $Pt(A)$. If $f : A \longrightarrow B$ is a **Loc**-morphism, then $Pt(f) : Pt(A) \longrightarrow Pt(B),\ p \mapsto p \circ f^{op}$ is a continuous map. Thus, Pt is a functor from **Loc** to **Top**. Thirdly, Ω is a left adjoint for Pt. Fourthly, for a topological space (X, τ), the map $\Psi : (X, \tau) \longrightarrow (Pt(\tau), \Phi^{\rightarrow}(\tau))$ given by $(\Psi(x))(U) = \chi_U(x)$ is continuous, where χ_U is the characteristic function of $U \in \tau$. Let **SpatLoc** be the full subcategory of **Loc** consisting of spatial locales—the locales with Φ being injective, and let **SobTop** be the full subcategory of **Top** consisting of sober spaces—the topological spaces with Ψ being bijective. Then $\Omega \dashv Pt$ restricts to a categorical equivalence between **SobTop** and **SpatLoc**.

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E-mail address: yaowei0516@163.com.

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Inspired by the notion of fuzzy topology of Chang [4], many researchers have successfully generalized the theory of general topology to the fuzzy setting. Given a topological space, the family of all open sets provides a frame under set inclusion. A fuzzy topology, based on a frame is also a frame under pointwise order (cf. Section 2.1). In fact, this point of view does not capture the fuzzy/order structure of the fuzzy topology and also makes no difference between fuzzy and crisp topology, unless the former uses the concept of semi-quantale, introduced by Rodabaugh [32,33] as a suitable algebraic structure for lattice-valued topology.

In order to study the lattice-theoretic structure of fuzzy topology without losing the fuzzy standpoint, we may firstly define a kind of fuzzy frame with fuzzy topology as examples. At the moment, there are about three kinds of poslat (point-set lattice-theoretic) fuzzy frames: Zhang–Liu-L-frames [47] as a comma category of classical frames, Pultr–Rodabaugh-L-frames [25,27] defined using categorical language and Yao-L-frames [43] defined through L-ordered sets.

This paper is organized as follows. In Section 2, we give a survey of fuzzifications of frames, the Papert–Papert–Isbell adjunction and related sobriety including the above-mentioned approaches. In Section 3, we establish categorical isomorphisms between Yao-L-frames, Zhang–Liu-L-frames and L-algebras. In Section 4, we study relations between L-sobriety, modified L-sobriety and $\imath_L$-sobriety. In Section 5, we study the properties of modified L-sobriety. Results in this section indicate that modified L-sobriety is the most reasonable fuzzy sobriety.

Let us recall some preliminaries which will be used throughout the paper.

For category theory, we recommend [2,20]. A pair of functors (F, G) is called an adjunction between categories $\mathbf{A}$ and $\mathbf{B}$, in symbols $F \dashv G$, if for every $A \in ob(\mathbf{A})$, $B \in ob(\mathbf{B})$, there exists a bijection between $hom_{\mathbf{A}}(A, G(B))$ and $hom_{\mathbf{B}}(F(A), B)$, which is natural in $\mathbf{A}$ and $\mathbf{B}$. The functor F is called a left adjoint of G and G a right adjoint of F. A category $\mathbf{A}$ is called dually isomorphic to (dual to, for short) a category $\mathbf{B}$ if $\mathbf{A}$ and $\mathbf{B}^{op}$ are isomorphic, i.e., there exist two functors $F : \mathbf{A} \longrightarrow \mathbf{B}^{op}$ and $G : \mathbf{B}^{op} \longrightarrow \mathbf{A}$ such that $F \circ G = \mathrm{id}_{\mathbf{B}^{op}}$ and $G \circ F = \mathrm{id}_{\mathbf{A}}$. Such an environment is also called a duality between $\mathbf{A}$ and $\mathbf{B}$.

Given an object B of the category $\mathbf{C}$, $(B \downarrow \mathbf{C})$ is the category of $\mathbf{C}$-objects under B with objects all pairs (f, C), where C is an object of $\mathbf{C}$ and $f : B \longrightarrow C$ is a morphism of $\mathbf{C}$, and with morphisms $h : (f, C) \longrightarrow (f', C')$ those morphisms $h : C \longrightarrow C'$ of $\mathbf{C}$ for which $h \circ f = f'$. The category $(\mathbf{C} \downarrow B)$ of objects over B is defined dually. Both $(B \downarrow \mathbf{C})$ and $(\mathbf{C} \downarrow B)$ are special cases of comma categories.

For lattices, fuzzy topology and quantales, we recommend [12,31,35]. A poset is called a complete lattice provided that joins (or suprema) and meets (or infima) of all its subsets exist. A complete lattice is sometimes called a sup-lattice (see, e.g., [35]) since the existence of joins of all subsets is equivalent to the existence of meets of all subsets. Given a complete lattice C, we use $\top_C$, $\bot_C$ to denote the top and the bottom element respectively.

A quantale is a triple $(Q, \leq, \&)$ such that (i) $(Q, \leq)$ is a complete lattice; (ii) $(Q, \&)$ is a semigroup; (iii) $q\&(\bigvee S) = \bigvee_{s\in S}(q\&s)$ and $(\bigvee S)\&q = \bigvee_{s\in S}(s\&q)$ for every $q \in Q$ and every $S \subseteq Q$. A quantale Q is said to be unital provided that there exists an element 1_Q (called the unit) such that $(Q, \&, 1_Q)$ is a monoid. Q is said to be commutative provided that the semigroup $(Q, \&)$ is commutative. A map $f : Q_1 \longrightarrow Q_2$ between quantales is called a quantale homomorphism if f is a semigroup homomorphism (i.e., $f(p\&q) = f(p)\&f(q)$ for every $p, q \in Q_1$) and join-preserving (i.e., $f(\bigvee S) = \bigvee_{s\in S} f(s)$ for every $S \subseteq Q_1$). A quantale homomorphism $f : Q_1 \longrightarrow Q_2$ between unital quantales is called unital if f preserves the unit (i.e., $f(1_{Q_1}) = 1_{Q_2}$). The category of quantales and quantale homomorphisms is denoted **Quant** and its subcategory of unital quantales and unital quantale homomorphisms is denoted **UQuant**.

A quantale $(L, \leq, \&)$ is called a frame or a complete Heyting algebra if $\& = \wedge$. In this case, the unit is the top element $\top_L$. For a frame L, it is well-known that there exists a derived operation $\rightarrow: L \times L \longrightarrow L$ satisfying $a \wedge b \leq c$ iff $a \leq b \rightarrow c$, for every $a, b, c \in L$. A map $f : A \longrightarrow B$ between frames is called a frame homomorphism if f preserves finite meets and arbitrary joins (which implies $f(\bot_A) = \bot_B,\ f(\top_A) = \top_B$). The resulting category is denoted **Frm** and its dual **Loc**.

An element $a \neq \top_L$ in L is called prime if $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$ for every $b, c \in L$. The lattice L is called spatial provided that every element in L can be represented as a meet of prime elements.

A complete lattice L is said to be completely distributive if it satisfies the complete distributivity law, i.e.,

$$\text{(CD1)} \quad \bigvee_{i\in I}\left(\bigwedge_{j\in J_i} a_i^j\right) = \bigwedge_{f\in\prod_{i\in I} J_i}\left(\bigvee_{i\in I} a_i^{f(i)}\right)$$