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ScienceDirect
Fuzzy Sets and Systems 161 (2010) 1064 – 1077
FUZZY sets and systems
www.elsevier.com/locate/fss

Characterization of the sendograph-convergence of fuzzy sets by means of their L_p- and levelwise convergence

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Received 19 August 2008; received in revised form 24 July 2009; accepted 29 July 2009
Available online 5 August 2009

Abstract

One particular way to define a metric on the family of all d-dimensional fuzzy sets with non-empty convex compact α-cuts is the sendograph metric which measures the Hausdorff distance of the sendographs of fuzzy sets. As first result it will be proved that if a sequence of fuzzy sets converges to a limit fuzzy set with respect to the sendograph metric then it also converges with respect to the L_p-metric and that the converse is true if in addition the supports of the sequence converge to the support of the limit. Additionally, convergence with respect to the sendograph metric will be shown to be equivalent to almost everywhere convergence of the α-cuts of the sequence to the α-cut of the limit plus the convergence of the supports of the sequence to the support of the limit (already proved in the one-dimensional setting by Fan). As second and stronger result it will be proved that analogous convergence interrelations also hold for the bigger class of fuzzy sets not necessarily having compact support if the Hausdorff distance is replaced by a metrization of the Fell topology. This in particular implies that the two characterizations proved in the first step do not depend on the choice of the metric used in the definition of the Hausdorff metric. In addition, it is demonstrated that the characterization of weakly compact sets with respect to the sendograph metric as stated by Greco is a straightforward consequence of the abovementioned equivalences and an alternative definition of fuzzy random variables that is equivalent to the common definition is given.
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Keywords: Fuzzy sets; Compact sets; Sendograph metric; L_p-metric; Fell topology; Myopic topology

1. Introduction

One possible choice of a metric on the family $\mathcal{F}_{c,c}(\mathbb{R}^d)$ of all d-dimensional fuzzy sets with non-empty convex compact α-cuts for every $\alpha \in [0, 1]$ is the so-called sendograph metric D_{send}, which measures the Hausdorff distance of the sendographs (supported endographs) of fuzzy sets. In [5] some properties of the sendograph metric were studied and the interrelation with other common metrics was analyzed. Unfortunately not all results presented [5] are correct, in particular the characterization of relative-compactness in $(\mathcal{F}_{c,c}(\mathbb{R}^d), D_{send})$ is wrong, a correction in the univariate setting was given in [7]. In [9] the characterization was generalized to fuzzy subsets of arbitrary metric spaces.

One question that naturally arises in connection with the sendograph metric D_{send} is whether there exists an easy and practicable characterization of the D_{send}-convergence on $\mathcal{F}_{c,c}(\mathbb{R}^d)$ by means of the convergence of the corresponding

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doi:10.1016/j.fss.2009.07.005

α-cuts. It follows directly from [16] that given a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{F}_{c,c}(\mathbb{R}^d)$ and a fuzzy set $u \in \mathcal{F}_{c,c}(\mathbb{R}^d)$ convergence of the α-cut of u_n to the α-cut of u for every $\alpha \in \mathbb{Q} \cap [0, 1]$ implies convergence w.r.t. D_{send}.

In this paper it will be proved that D_{send}-convergence of a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{F}_{c,c}(\mathbb{R}^d)$ to $u \in \mathcal{F}_{c,c}(\mathbb{R}^d)$ is equivalent to almost everywhere convergence of the α-cut of u_n to the α-cut of u (as function in α) plus the convergence of the support of u_n to the support of u (both with respect to the Hausdorff metric). As a consequence D_{send}-convergence on $\mathcal{F}_{c,c}(\mathbb{R}^d)$ is seen to be equivalent to L_p-convergence of u_n to u plus convergence of the support of u_n to the support of u. For dimension $d = 1$ these interrelations have already been proved in [8,11]. Nevertheless, since in the multivariate setting at least additionally it has to be shown that the function $\alpha \mapsto [u]_\alpha$ has only countably many discontinuities, which can be done using support functions, we will prove the two characterizations mentioned above mainly working with support functions and do not try to extend the proof given in [8] (Section 3).

Moreover, and more importantly, in order to prove a convergence result that does not depend on the concrete metric in $\mathbb{R}^d$ used in the definition of the Hausdorff metric but only on the topology of $\mathbb{R}^d$ it will be shown that the same two convergence interrelations also hold if we replace the Hausdorff metric by a metrization of the Fell (hit-and-miss) topology (see [15,19]). Considering the Fell topology also allows us to consider the bigger class $\mathcal{F}_c(\mathbb{R}^d)$ of all d-dimensional fuzzy sets not necessarily having compact support and prove the analogous two characterizations on $\mathcal{F}_c(\mathbb{R}^d)$ (Section 4). As small applications it will be demonstrated that the characterizations of weakly compact sets in the metric space $(\mathcal{F}_{c,c}(\mathbb{R}^d), D_{send})$ given by [7,9] are straightforward consequences of the abovementioned two equivalences. In addition, an alternative definition of fuzzy random variables by means of measurability with respect to D_{send} is stated and shown to be equivalent to the common definitions.

2. Notation and preliminaries

Throughout the whole paper the i-th coordinate of a vector $x \in \mathbb{R}^d$ is written as $\pi_i(x)$ for every $i \in \{1, \dots, d\}$ in order to clearly distinguish between coordinates and indices of sequences. $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^d$, $\langle\cdot,\cdot\rangle$ the Euclidean inner product on $\mathbb{R}^d$,

$$B(A, r) := \{y \in \mathbb{R}^d : \exists x \in A \text{ such that } \|x - y\|_2 < r\}$$

the open r-neighbourhood of A ($r > 0$), $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ the unit sphere in $\mathbb{R}^d$. In case of $A = \{z\}$ we will write $B(z, r)$ for the open ball of radius r around z. $\mathcal{C}(\mathbb{R}^d)$ denotes the family of all closed subsets of $\mathbb{R}^d$, $\mathcal{K}(\mathbb{R}^d)$ the family of all non-empty compact subsets of $\mathbb{R}^d$ and $\mathcal{K}_c(\mathbb{R}^d)$ the family of all convex elements in $\mathcal{K}(\mathbb{R}^d)$. The *Hausdorff metric* on $\mathcal{K}(\mathbb{R}^d)$ will be denoted by δ_H—since no confusion will arise the symbol δ_H will be used for every dimension d.

It is well-known that firstly $(\mathcal{K}(\mathbb{R}^d), \delta_H)$ is a complete metric space in which every bounded set is weakly compact and that the limit $A \in \mathcal{K}(\mathbb{R}^d)$ of a sequence $A_n \in \mathcal{K}(\mathbb{R}^d)$ can be expressed as (see [1])

$$A := \left\{x \in \mathbb{R}^d : \exists (x_n)_{n\in\mathbb{N}} \text{ such that } x_n \in A_n \forall n \text{ and } \lim_{n\to\infty} x_n = x\right\}. \tag{1}$$

And secondly that given a Cauchy sequence $(A_n)_{n\in\mathbb{N}}$ in $(\mathcal{K}(\mathbb{R}^d), \delta_H)$, a strictly monotonically increasing sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ and a Cauchy sequence $(x_{n_i})_{i\in\mathbb{N}}$ in $\mathbb{R}^d$ with $x_{n_i} \in A_{n_i}$ $\forall i$ there exists a Cauchy sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ fulfilling both $\tilde{x}_n \in A_n$ $\forall n$ and $\tilde{x}_{n_i} = x_{n_i}$ $\forall i$ (sometimes referred to as Extension Lemma). In fact these two properties imply that the limit A of a convergent sequence $(A_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{K}(\mathbb{R}^d)$ coincides with the topological limit of the sequence. For further properties of the Hausdorff metric see [1,14,18].

The *Fell* (*or hit-and-miss*) *topology* τ_F on $\mathcal{C}(\mathbb{R}^d)$ is defined as the topology generated by the base $\mathcal{B}$ consisting of all sets of the form (since no confusion will arise the symbol τ_F will be used for every dimension d)

$$\mathcal{C}^K_{G_1,\dots,G_n} := \{F \in \mathcal{C}(\mathbb{R}^d) : F \cap K = \emptyset, F \cap G_1 \neq \emptyset, \dots, F \cap G_n \neq \emptyset\}, \tag{2}$$

whereby $K \subseteq \mathbb{R}^d$ is compact, $n \geq 0$, and $G_1, \dots, G_n$ are open subsets of $\mathbb{R}^d$ (see [14,15,19]). Since $\mathbb{R}^d$ is a locally compact Polish space $(\mathcal{C}(\mathbb{R}^d), \tau_F)$ is a compact, second countable Hausdorff space. Therefore τ_F is metrizable (see [14,15]). One possible metrization ρ of τ_F can be constructed by considering an inverse stereographic projection $\varphi_d : \mathbb{R}^d \to B_d$, whereby $B_d := \{x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d-1} \pi_i(x)^2 + (\pi_d(x) - 1)^2 + (\pi_{d+1}(x) - 1)^2 = 1\}$, defining

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