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The Hausdorff fuzzy quasi-metric

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Abstract

Removing the condition of symmetry in the notion of a fuzzy (pseudo)metric, in Kramosil and Michalek's sense, one has the notion of a fuzzy quasi-(pseudo-)metric. Then for each fuzzy quasi-pseudo-metric on a set X we construct a fuzzy quasi-pseudo-metric on the collection of all nonempty subsets of X, called the Hausdorff fuzzy quasi-pseudo-metric. We investigate several properties of this structure and present several illustrative examples as well as an application to the domain of words. The notion of Hausdorff fuzzy quasi-pseudo-metric when quasi-pseudo-metric fuzziness is considered in the sense of George and Veeramani is also discussed.

Keywords: The Hausdorff fuzzy quasi-pseudo-metric; Complete; Precompact; Preorder; The domain of words

1. Introduction

It is well known that the Hausdorff distance has an undoubted importance not only in general topology but also in other areas of Mathematics and Computer Science, such as convex analysis and optimization [6,31,37], dynamical systems [11,34,47,61], mathematical morphology [56], fractals [3,12], image processing [22,30,55,63], programming language and semantics [4,5], computational biology [21,57], etc. In [13], Egbert extended the classical construction of the Hausdorff distance of a metric space to Menger spaces. Later on, Tardiff [59] (see also [52,54]), generalized Egbert's construction to probabilistic metric spaces, obtaining in this way a suitable notion of a Hausdorff probabilistic distance. Since fuzzy metric spaces, in the sense of Kramosil and Michalek, are closely related to Menger spaces [23], one can easily define, from Egbert–Tardiff's construction, a Hausdorff fuzzy distance for a given fuzzy metric space. In connection with these constructions, a notion of Hausdorff fuzzy metric for fuzzy metric spaces in the sense of George and Veeramani [17,18] was discussed in [40].

On the other hand, it is well known that several structures of asymmetric topology like quasi-uniformities and (fuzzy) quasi-metrics, constitute efficient tools to formulate and solve problems in hyperspaces, function spaces, topological algebra, asymmetric functional analysis, point-free geometry, complexity of algorithms, theoretical computer science, etc. (see, for instance, [25, Chapters 11 and 12, 26, Section 3], and also [1,2,10,16,19,27,32,41–43,46,49,53,62], etc. for recent contributions).

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In this paper we introduce and study notions of Hausdorff fuzzy quasi-metric (in the senses of Kramosil and Michalek, and George and Veeramani, respectively) that generalize to the asymmetric setting the corresponding notions of Hausdorff fuzzy metric. In this way, we partially reconcile the theory of fuzzy metric hyperspaces with the theory of asymmetric topology. Furthermore, we apply our approach to the domain of words, a paradigmatic example of a space that naturally appears in the theory of computation.

The paper is organized as follows. In Section 2 we present the basic notions and results which will need later on. In Section 3 we construct and discuss a notion of Hausdorff fuzzy quasi-metric, based on the notion of fuzzy (quasi-)metric of Kramosil and Michalek. In Section 4 we shall show that this new concept has several nice properties of completeness, precompactness and compactness. In Section 5 we consider a notion of Hausdorff fuzzy quasi-metric, based on the notion of fuzzy (quasi-)metric in the sense of George and Veeramani. In Section 6 we apply the theory developed in the preceding sections to the domain of words and we point out some advantages of the use of fuzzy quasi-metrics instead of classical metrics and quasi-metrics. Finally, we present our conclusions.

2. Basic notions and preliminary results

In the sequel the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of real numbers, the set of nonnegative real numbers and the set of positive integer numbers, respectively.

Our basic references for quasi-metric spaces and quasi-uniform spaces are [15,25], and for general topology it is [14].

Let us recall that a quasi-uniformity on a set X is a filter $\mathcal{U}$ on $X \times X$ such that:

(i) for each $U \in \mathcal{U}$, $\Delta \subseteq U$, where $\Delta = \{(x, x) : x \in X\}$;
(ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$, where $V^2 = \{(x, y) \in X \times X :$ there is $z \in X$ with $(x, z) \in V$ and $(z, y) \in V\}$.

By a quasi-uniform space we mean a pair $(X, \mathcal{U})$ such that X is a nonempty set and $\mathcal{U}$ is a quasi-uniformity on X.

Each quasi-uniformity $\mathcal{U}$ on X generates a topology $\tau_{\mathcal{U}}$ on X such that a neighborhood base for each point $x \in X$ is given by $\{U(x) : U \in \mathcal{U}\}$, where $U(x) = \{y \in X : (x, y) \in \mathcal{U}\}$.

Given a quasi-uniformity $\mathcal{U}$ on X, then the filter $\mathcal{U}^{-1}$ defined on $X \times X$ by $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is also a quasi-uniformity on X, called the conjugate of $\mathcal{U}$, and the filter $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$ is a uniformity on X. (As usual, $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$.)

An extended quasi-pseudo-metric on a set X is a function $d : X \times X \to [0, +\infty]$ such that for all $x, y, z \in X$:

(i) $d(x, x) = 0$;
(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

Following the modern terminology (see, for instance, [25, Chapter 11]), an extended quasi-metric on X is an extended quasi-pseudo-metric d on X which satisfies the condition:

(i′) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$.

An extended quasi-(pseudo-)metric d on X such that $d(x, y) < +\infty$ for all $x, y \in X$, is said to be a quasi-(pseudo-)metric on X.

By a quasi-(pseudo-)metric space we mean a pair (X, d) such that X is a nonempty set and d is a quasi-(pseudo-)metric on X.

The following is an easy but paradigmatic example of a quasi-metric space.

Example 1. Let ℓ be the function defined on $\mathbb{R} \times \mathbb{R}$ by $\ell(x, y) = \max\{x - y, 0\}$. Then ℓ is a quasi-metric on $\mathbb{R}$ such that ℓ^s is the Euclidean metric on $\mathbb{R}$.

Each extended quasi-pseudo-metric d on X generates a topology τ_d on X which has as a base the family of open balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Observe that if d is an extended quasi-metric on X, then τ_d is a T_0 topology on X.

A topological space (X, τ) is said to be quasi-(pseudo-)metrizable if there is a quasi-(pseudo-)metric d on X such that $\tau = \tau_d$.